

VECTORS (Q 2, PAPER 2)

LESSON NO. 3: DOT PRODUCT

2006

2 (b) $\vec{p} = -5\vec{i} + 2\vec{j}$, $\vec{q} = \vec{i} - 6\vec{j}$ and $\vec{r} = -\vec{i} + 5\vec{j}$.

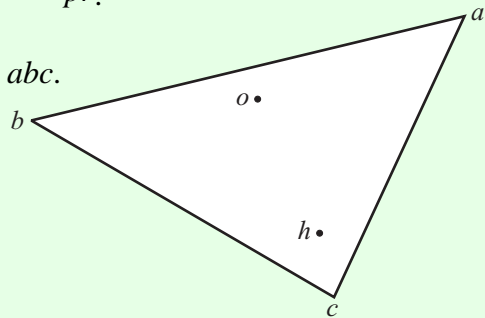
(i) Express \vec{pq} and \vec{pr} in terms of \vec{i} and \vec{j} .

(ii) Given that $10\vec{s} = |\vec{pr}|\vec{pq} + |\vec{pq}|\vec{pr}$, express \vec{s} in terms of \vec{i} and \vec{j} .

(iii) Find the measure of the angle between \vec{s} and \vec{pr} .

2 (c) The origin o is the circumcentre of the triangle abc .

If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$, show that $\vec{ah} \perp \vec{bc}$.



$\vec{ab} = \vec{b} - \vec{a}$ 1

SOLUTION

2 (b) (i)

$\vec{pq} = \vec{q} - \vec{p} = (\vec{i} - 6\vec{j}) - (-5\vec{i} + 2\vec{j}) = 6\vec{i} - 8\vec{j}$

$\vec{pr} = \vec{r} - \vec{p} = (-\vec{i} + 5\vec{j}) - (-5\vec{i} + 2\vec{j}) = 4\vec{i} + 3\vec{j}$

2(b) (ii)

$|\vec{pr}| = |4\vec{i} + 3\vec{j}| = \sqrt{16+9} = \sqrt{25} = 5$

$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$ 5

$|\vec{pq}| = |6\vec{i} - 8\vec{j}| = \sqrt{36+64} = \sqrt{100} = 10$

$\therefore 10\vec{s} = |\vec{pr}|\vec{pq} + |\vec{pq}|\vec{pr} \Rightarrow 10\vec{s} = 5(6\vec{i} - 8\vec{j}) + 10(4\vec{i} + 3\vec{j})$

$\Rightarrow 10\vec{s} = 30\vec{i} - 40\vec{j} + 40\vec{i} + 30\vec{j} \Rightarrow 10\vec{s} = 70\vec{i} - 10\vec{j}$

$\Rightarrow \vec{s} = 7\vec{i} - \vec{j}$

2 (b) (iii)

$\vec{s} \cdot \vec{pr} = |\vec{s}||\vec{pr}|\cos\theta$

$\vec{ab} \cdot \vec{ac} = |\vec{ab}||\vec{ac}|\cos\theta$ 8

$\Rightarrow (7\vec{i} - \vec{j}) \cdot (4\vec{i} + 3\vec{j}) = |7\vec{i} - \vec{j}||4\vec{i} + 3\vec{j}|\cos\theta$

$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = 0$ 9

$\Rightarrow 25 = \sqrt{50}\sqrt{25}\cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$

$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = |\vec{i}|^2 = |\vec{j}|^2 = 1$ 10

TRICK: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients.

2 (c)

DOT PRODUCT PROPERTIES

1. $\vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{a}$ [You can switch the order of dot product]
2. $\theta = 90^\circ \Leftrightarrow \vec{a} \cdot \vec{c} = |\vec{a}| |\vec{c}| \cos 90^\circ = 0$
 [If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]
3. $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \Leftrightarrow \vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| \cos 0^\circ = |\vec{a}|^2$
 [If you dot product a vector with itself you get the modulus squared. The modulus squared of a vector is got by dotting it with itself.]

If o is the circumcentre of triangle abc , o is equidistant from a , b and c .

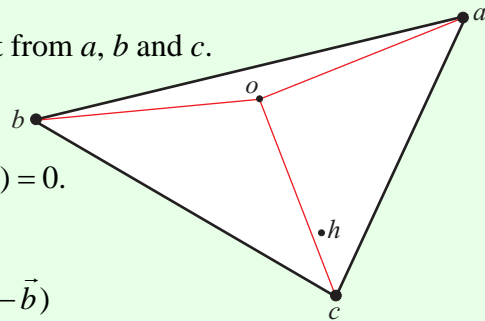
$$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$$

To prove that $\vec{ah} \perp \vec{bc}$, you need to show that $(\vec{ah}) \cdot (\vec{bc}) = 0$.

$$(\vec{ah}) \cdot (\vec{bc}) = (\vec{h} - \vec{a}) \cdot (\vec{c} - \vec{b})$$

But $\vec{h} = \vec{a} + \vec{b} + \vec{c} \Rightarrow (\vec{a} + \vec{b} + \vec{c} - \vec{a}) \cdot (\vec{c} - \vec{b}) = (\vec{b} + \vec{c}) \cdot (\vec{c} - \vec{b})$

$$= \vec{b} \cdot \vec{c} - \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{b} = |\vec{c}|^2 - |\vec{b}|^2 = 0$$



2005

2 (c) $\vec{u} = \vec{i} + 5\vec{j}$ and $\vec{v} = 4\vec{i} + 4\vec{j}$.

(i) Find $\cos \angle uov$, where o is the origin.

(ii) $\vec{r} = (1-k)\vec{u} + k\vec{v}$, where $k \in \mathbf{R}$ and $k \neq 0$. Find the value of k for which

$$|\angle uov| = |\angle vor|.$$

SOLUTION

2 (c) (i)

$$\vec{ou} = \vec{i} + 5\vec{j}, \vec{ov} = 4\vec{i} + 4\vec{j}$$

$$\vec{a} \cdot \vec{c} = |\vec{a}| |\vec{c}| \cos \theta \dots\dots 8$$

$$\vec{ou} \cdot \vec{ov} = |\vec{ou}| |\vec{ov}| \cos \angle uov$$

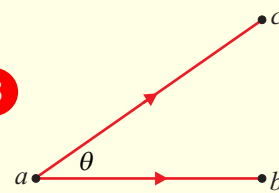
$$\Rightarrow (\vec{i} + 5\vec{j}) \cdot (4\vec{i} + 4\vec{j}) = |\vec{i} + 5\vec{j}| |4\vec{i} + 4\vec{j}| \cos \angle uov$$

$$\Rightarrow 24 = \sqrt{26} \sqrt{32} \cos \angle uov \Rightarrow \cos \angle uov = \frac{24}{4\sqrt{26}\sqrt{2}} = \frac{3}{\sqrt{13}}$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = 0 \dots\dots 9$$

TRICK: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients.

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2} \dots\dots 5$$



2 (c) (ii)

$$\vec{or} = (1-k)\vec{u} + k\vec{v}$$

$$\Rightarrow \vec{or} = (1-k)(\vec{i} + 5\vec{j}) + k(4\vec{i} + 4\vec{j}) = (3k+1)\vec{i} + (5-k)\vec{j}$$

$$\text{If } |\angle uov| = |\angle vor| \Rightarrow \cos |\angle uov| = \cos |\angle vor| \Rightarrow \cos |\angle vor| = \frac{3}{\sqrt{13}}$$

$$\vec{or} \cdot \vec{ov} = |\vec{or}| |\vec{ov}| \cos \angle vor \Rightarrow \cos \angle vor = \frac{\vec{or} \cdot \vec{ov}}{|\vec{or}| |\vec{ov}|}$$

$$\Rightarrow \frac{3}{\sqrt{13}} = \frac{[(3k+1)\vec{i} + (5-k)\vec{j}] \cdot [4\vec{i} + 4\vec{j}]}{|(3k+1)\vec{i} + (5-k)\vec{j}| |4\vec{i} + 4\vec{j}|} \Rightarrow \frac{3}{\sqrt{13}} = \frac{4(3k+1) + 4(5-k)}{\sqrt{(3k+1)^2 + (5-k)^2} \sqrt{32}}$$

$$\Rightarrow \frac{3}{\sqrt{13}} = \frac{12k + 4 + 20 - 4k}{4\sqrt{(3k+1)^2 + (5-k)^2} \sqrt{2}} \Rightarrow \frac{3}{\sqrt{13}} = \frac{8k + 24}{4\sqrt{(3k+1)^2 + (5-k)^2} \sqrt{2}}$$

$$\Rightarrow \frac{3}{\sqrt{13}} = \frac{2k + 6}{\sqrt{(3k+1)^2 + (5-k)^2} \sqrt{2}} \Rightarrow 3\sqrt{(3k+1)^2 + (5-k)^2} \sqrt{2} = (2k + 6)\sqrt{13}$$

$$\Rightarrow 18[(3k+1)^2 + (5-k)^2] = 13(2k+6)^2$$

$$\Rightarrow 18[10k^2 - 4k + 26] = 13(4k^2 + 24k + 36)$$

$$\Rightarrow 180k^2 - 72k + 468 = 52k^2 + 312k + 468$$

$$\Rightarrow 128k^2 - 384k = 0 \Rightarrow k^2 - 3k = 0 \Rightarrow k(k-3) = 0$$

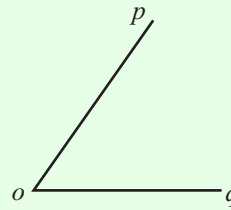
$$\Rightarrow k = 3. \text{ (You were told that } k \neq 0. \text{)}$$

2004

2 (c) p and q are points and o is the origin. \vec{p} , \vec{q} and \vec{o} are not collinear and $|\vec{p}| = |\vec{q}|$.

(i) Prove that \vec{pq} is perpendicular to $(\vec{p} + \vec{q})$.

(ii) Prove that $\vec{po} \cdot \vec{pq} = \frac{1}{2} |\vec{pq}|^2$.



SOLUTION

2 (c) (i)

DOT PRODUCT PROPERTIES

1. $\vec{ab} \cdot \vec{ac} = \vec{ac} \cdot \vec{ab}$ [You can switch the order of dot product]

2. $\theta = 90^\circ \Leftrightarrow \vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos 90^\circ = 0$

[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

3. $\vec{ab} = \vec{ac} \Leftrightarrow \vec{ab} \cdot \vec{ab} = |\vec{ab}| |\vec{ab}| \cos 0^\circ = |\vec{ab}|^2$

[If you dot product a vector with itself you get the modulus squared. The modulus squared of a vector is got by dotting it with itself.]

$$\vec{pq} \cdot (\vec{p} + \vec{q}) = (\vec{q} - \vec{p}) \cdot (\vec{p} + \vec{q}) = \vec{q} \cdot \vec{p} + \vec{q} \cdot \vec{q} - \vec{p} \cdot \vec{p} - \vec{p} \cdot \vec{q}$$

$$= |\vec{q}|^2 - |\vec{p}|^2 = 0 \text{ as you are told that } |\vec{p}| = |\vec{q}|.$$

Therefore, \vec{pq} is perpendicular to $(\vec{p} + \vec{q})$ as their dot product is zero.

2 (c) (ii)

$$\vec{po} \cdot \vec{pq} = -\vec{p} \cdot (\vec{q} - \vec{p}) = \vec{p} \cdot \vec{p} - \vec{p} \cdot \vec{q} = |\vec{p}|^2 - \vec{p} \cdot \vec{q} \text{ (LHS)}$$

$$\frac{1}{2} |\vec{pq}|^2 = \frac{1}{2} \vec{pq} \cdot \vec{pq} = \frac{1}{2} (\vec{q} - \vec{p}) \cdot (\vec{q} - \vec{p}) = \frac{1}{2} (\vec{q} \cdot \vec{q} - 2\vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{p})$$

$$= \frac{1}{2} (|\vec{q}|^2 - 2\vec{p} \cdot \vec{q} + |\vec{p}|^2) = \frac{1}{2} (-2\vec{p} \cdot \vec{q} + 2|\vec{p}|^2) = -\vec{p} \cdot \vec{q} + |\vec{p}|^2 \text{ (RHS)}$$

2003

2 (b) $\vec{p} = 2\vec{i} + \vec{j}$, $\vec{q} = 3\vec{i} + k\vec{j}$, $\vec{r} = 3\vec{i} + t\vec{j}$ where $k, t \in \mathbf{R}$ and o is the origin.

(i) Given that $\vec{p} \perp \vec{q}$, calculate the value of k .

(ii) Given that $|\angle por| = 45^\circ$, calculate the two possible values of t .

SOLUTION

2 (b) (i)

$$2. \quad \theta = 90^\circ \Leftrightarrow \vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos 90^\circ = 0$$

[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

$$\begin{aligned} \text{If } \vec{p} \perp \vec{q} &\Rightarrow \vec{p} \cdot \vec{q} = 0 \Rightarrow (2\vec{i} + \vec{j}) \cdot (3\vec{i} + k\vec{j}) = 0 \\ &\Rightarrow 6 + k = 0 \Rightarrow k = -6 \end{aligned}$$

TRICK: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients.

2 (b) (ii)

$$\vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos \theta \quad \dots\dots \quad \mathbf{8}$$

$$\vec{op} \cdot \vec{or} = |\vec{op}| |\vec{or}| \cos 45^\circ$$

$$\Rightarrow (2\vec{i} + \vec{j}) \cdot (3\vec{i} + t\vec{j}) = |2\vec{i} + \vec{j}| |3\vec{i} + t\vec{j}| \left(\frac{1}{\sqrt{2}}\right)$$

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2} \quad \dots\dots \quad \mathbf{5}$$

$$\Rightarrow 6 + t = \frac{\sqrt{5}\sqrt{9+t^2}}{\sqrt{2}} \Rightarrow \sqrt{2}(6+t) = \sqrt{5}\sqrt{9+t^2}$$

$$\Rightarrow 2(6+t)^2 = 5(9+t^2) \Rightarrow 2(36+12t+t^2) = 45+5t^2$$

$$\Rightarrow 72+24t+2t^2 = 45+5t^2 \Rightarrow 3t^2 - 24t - 27 = 0$$

$$\Rightarrow t^2 - 8t - 9 = 0 \Rightarrow (t-9)(t+1) = 0 \Rightarrow t = -1, 9$$

2002

2 (c) $\vec{k} = \vec{i} + 3\vec{j}$, $\vec{n} = 4\vec{i} - 2\vec{j}$, $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = x\vec{i} + y\vec{j}$ where $x, y \in \mathbf{R}$.

(i) Express the value of $\vec{kn} \cdot \vec{kv}$ in the form $ax + by + c$ where $a, b, c \in \mathbf{R}$.

(ii) Prove that if $\vec{kn} \cdot \vec{kv} = \vec{kn} \cdot \vec{ku}$, and $\vec{u} \neq \vec{v}$, then $\vec{kn} \perp \vec{uv}$.

SOLUTION

2 (c) (i)

$$\begin{aligned}\vec{kn} \cdot \vec{kv} &= (\vec{n} - \vec{k}) \cdot (\vec{v} - \vec{k}) = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}) \cdot (x\vec{i} + y\vec{j} - \vec{i} - 3\vec{j}) \\ &= (3\vec{i} - 5\vec{j}) \cdot ((x-1)\vec{i} + (y-3)\vec{j}) = 3(x-1) - 5(y-3) \\ &= 3x - 3 - 5y + 15 = 3x - 5y + 12\end{aligned}$$

2 (c) (ii)

$$\text{If } \vec{kn} \cdot \vec{kv} = \vec{kn} \cdot \vec{ku} \Rightarrow 3x - 5y + 12 = (\vec{n} - \vec{k}) \cdot (\vec{u} - \vec{k})$$

$$\Rightarrow 3x - 5y + 12 = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}) \cdot (2\vec{i} + \vec{j} - \vec{i} - 3\vec{j})$$

$$\Rightarrow 3x - 5y + 12 = (3\vec{i} - 5\vec{j}) \cdot (\vec{i} - 2\vec{j}) = 13 \Rightarrow 3x - 5y - 1 = 0 \dots \text{(1)}$$

TRICK: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients.

$$2. \quad \theta = 90^\circ \Leftrightarrow \vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos 90^\circ = 0$$

[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

You can prove that $\vec{kn} \perp \vec{uv}$ by showing that their dot product is zero.

$$\begin{aligned}\vec{kn} \cdot \vec{uv} &= (\vec{n} - \vec{k}) \cdot (\vec{v} - \vec{u}) = (4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j}) \cdot (x\vec{i} + y\vec{j} - 2\vec{i} - \vec{j}) \\ &= (3\vec{i} - 5\vec{j}) \cdot ((x-2)\vec{i} + (y-1)\vec{j}) = 3(x-2) - 5(y-1) \\ &= 3x - 6 - 5y + 5 = 3x - 5y - 1 = 0 \quad \text{(From equation (1)).}\end{aligned}$$

2001

2 (c) rst is a triangle where $\vec{r} = -\vec{i} + 2\vec{j}$, $\vec{s} = -4\vec{i} - 2\vec{j}$ and $\vec{t} = 3\vec{i} - \vec{j}$.

(i) Express \vec{rs} , \vec{st} and \vec{tr} in terms of \vec{i} and \vec{j} .

(ii) Show that the triangle rst is right-angled at r .

(iii) Find the measure of $\angle rst$.

SOLUTION

2 (c) (i)

$$\vec{rs} = \vec{s} - \vec{r} = -4\vec{i} - 2\vec{j} + \vec{i} - 2\vec{j} = -3\vec{i} - 4\vec{j}$$

$$\vec{st} = \vec{t} - \vec{s} = 3\vec{i} - \vec{j} + 4\vec{i} + 2\vec{j} = 7\vec{i} + \vec{j}$$

$$\vec{tr} = \vec{r} - \vec{t} = -\vec{i} + 2\vec{j} - 3\vec{i} + \vec{j} = -4\vec{i} + 3\vec{j}$$

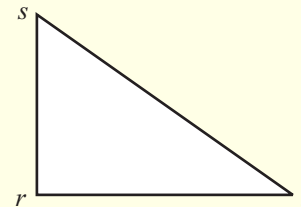
$$\vec{ab} = \vec{b} - \vec{a} \dots\dots \mathbf{1}$$

2 (c) (ii)

$$2. \theta = 90^\circ \Leftrightarrow \vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos 90^\circ = 0$$

[If two vectors are perpendicular their dot product is zero. If their dot product is zero, the vectors are perpendicular.]

TRICK: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients when doing the dot product..



You need to show that $\vec{sr} \cdot \vec{rt} = 0$.

$$\vec{rs} \cdot \vec{tr} = (-3\vec{i} - 4\vec{j}) \cdot (-4\vec{i} + 3\vec{j}) = 12 - 12 = 0$$

2 (c) (iii)

$$\vec{ab} \cdot \vec{ac} = |\vec{ab}| |\vec{ac}| \cos \theta \dots\dots \mathbf{8}$$

$$\vec{sr} \cdot \vec{st} = |\vec{sr}| |\vec{st}| \cos \angle rst \Rightarrow \cos \angle rst = \frac{\vec{sr} \cdot \vec{st}}{|\vec{sr}| |\vec{st}|}$$

$$\Rightarrow \cos \angle rst = \frac{(3\vec{i} + 4\vec{j}) \cdot (7\vec{i} + \vec{j})}{|3\vec{i} + 4\vec{j}| |7\vec{i} + \vec{j}|} \Rightarrow \cos \angle rst = \frac{21 + 4}{\sqrt{25} \sqrt{50}} = \frac{25}{5 \times 5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \angle rst = 45^\circ$$