

VECTORS (Q 2, PAPER 2)

2011

2. (a) Find the value of s and the value of t that satisfy the equation

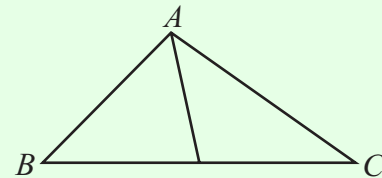
$$s(\vec{i} - 4\vec{j}) + t(2\vec{i} + 3\vec{j}) = 4\vec{i} - 27\vec{j}.$$

(b) $\overrightarrow{OP} = 3\vec{i} - 4\vec{j}$ and $\overrightarrow{OQ} = 5(\overrightarrow{OP})^\perp$, where O is the origin.

(i) Find \overrightarrow{OQ} in terms of \vec{i} and \vec{j} .

(ii) Find $\cos|\angle OQP|$, in surd form.

(c) ABC is a triangle and D is the mid-point of $[BC]$.



(i) Express \overrightarrow{AB} in terms of \overrightarrow{AD} and \overrightarrow{BC}
and express \overrightarrow{AC} in terms of \overrightarrow{AD} and \overrightarrow{BC} .

(ii) Hence, prove that $|AB|^2 + |AC|^2 = 2|AD|^2 + \frac{1}{2}|BC|^2$.

SOLUTION

2 (a)

ADDITION AND MULTIPLICATION BY A SCALAR (NUMBER)
Multiply out the brackets and add the \vec{i} 's together and the \vec{j} 's together.

EQUALITY
If $a\vec{i} + b\vec{j} = c\vec{i} + d\vec{j}$ then $a = c$ and $b = d$. Equate the \vec{i} 's and the \vec{j} 's.

$$s(\vec{i} - 4\vec{j}) + t(2\vec{i} + 3\vec{j}) = 4\vec{i} - 27\vec{j}$$

$$s\vec{i} - 4s\vec{j} + 2t\vec{i} + 3t\vec{j} = 4\vec{i} - 27\vec{j}$$

$$s\vec{i} + 2t\vec{i} - 4s\vec{j} + 3t\vec{j} = 4\vec{i} - 27\vec{j}$$

$$(s + 2t)\vec{i} + (-4s + 3t)\vec{j} = 4\vec{i} - 27\vec{j}$$

$$s + 2t = 4 \dots\dots (1)(\times 4)$$

$$-4s + 3t = -27 \dots (2)$$

$$4s + 8t = 16 \dots\dots (1)(\times 4)$$

$$-4s + 3t = -27 \dots (2)$$

$$11t = -11 \Rightarrow t = -1$$

Substitute this value of t into Eqn. (1): $s + 2(-1) = 4$

$$s - 2 = 4$$

$$\therefore s = 6$$

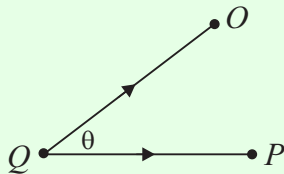
2 (b) (i)

$$\overrightarrow{OP} = 3\vec{i} - 4\vec{j}$$

$$\overrightarrow{OP}^\perp = 4\vec{i} + 3\vec{j} \quad \boxed{\vec{r} = +x\vec{i} + y\vec{j} \Rightarrow \vec{r}^\perp = -y\vec{i} + x\vec{j}}$$

$$\overrightarrow{OQ} = 5(\overrightarrow{OP}^\perp) = 5(4\vec{i} + 3\vec{j}) = 20\vec{i} + 15\vec{j}$$

2 (b) (ii)



Dot product: $\boxed{\overrightarrow{ab} \cdot \overrightarrow{ac} = |\overrightarrow{ab}| |\overrightarrow{ac}| \cos \theta}$

$$\overrightarrow{QO} \cdot \overrightarrow{QP} = |\overrightarrow{QO}| |\overrightarrow{QP}| \cos \theta$$

$$\overrightarrow{OQ} = -\overrightarrow{QO} = -20\vec{i} - 15\vec{j}$$

$$\overrightarrow{QP} = \vec{P} - \vec{Q} = (3\vec{i} - 4\vec{j}) - (20\vec{i} + 15\vec{j}) = -17\vec{i} - 19\vec{j} \quad \boxed{\overrightarrow{ab} = \vec{b} - \vec{a}}$$

$$\overrightarrow{QO} \cdot \overrightarrow{QP} = |\overrightarrow{QO}| |\overrightarrow{QP}| \cos \theta$$

$$(-20\vec{i} - 15\vec{j}) \cdot (-17\vec{i} - 19\vec{j}) = |-20\vec{i} - 15\vec{j}| |-17\vec{i} - 19\vec{j}| \cos \theta \quad \boxed{|\vec{r}| = \sqrt{x^2 + y^2}}$$

$$(-20)(-17) + (-15)(-19) = \sqrt{(-20)^2 + (-15)^2} \sqrt{(-17)^2 + (-19)^2} \cos \theta$$

$$625 = \sqrt{625} \sqrt{650} \cos \theta$$

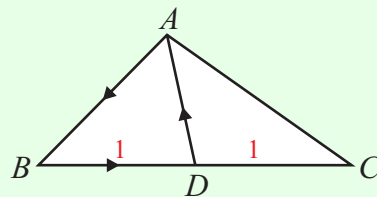
$$625 = 25 \times 5 \sqrt{26} \cos \theta$$

$$\therefore \cos \theta = \frac{5}{\sqrt{26}}$$

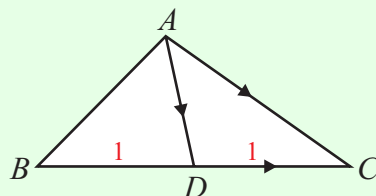
DOT PRODUCT OF $\{\vec{i}, \vec{j}\}$: Just multiply the \vec{i} coefficients and multiply the \vec{j} coefficients.

2 (c) (i)

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{BD} + \overrightarrow{DA} \\ &= \frac{1}{2} \overrightarrow{BC} - \overrightarrow{AD} \end{aligned}$$



$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AD} + \overrightarrow{DC} \\ &= \overrightarrow{AD} + \frac{1}{2} \overrightarrow{BC} \end{aligned}$$



2 (c) (ii)

DOT PRODUCT PROPERTIES

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AC} \cdot \overrightarrow{AB} \text{ [You can switch the order of dot product]}$$

$$|\overrightarrow{AB}|^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} \text{ [If you dot product a vector with itself you get the modulus squared.]}$$

The modulus squared of a vector is got by dotting it with itself.]

$$\begin{aligned} |\overrightarrow{AB}|^2 &= \overrightarrow{AB} \cdot \overrightarrow{AB} = \left(\frac{1}{2}\overrightarrow{BC} - \overrightarrow{AD}\right) \cdot \left(\frac{1}{2}\overrightarrow{BC} - \overrightarrow{AD}\right) \\ &= \frac{1}{4}\overrightarrow{BC} \cdot \overrightarrow{BC} - \frac{1}{2}\overrightarrow{BC} \cdot \overrightarrow{AD} - \frac{1}{2}\overrightarrow{AD} \cdot \overrightarrow{BC} + \overrightarrow{AD} \cdot \overrightarrow{AD} \\ &= \frac{1}{4}|\overrightarrow{BC}|^2 - \overrightarrow{BC} \cdot \overrightarrow{AD} + |\overrightarrow{AD}|^2 \end{aligned}$$

$$\begin{aligned} |\overrightarrow{AC}|^2 &= \overrightarrow{AC} \cdot \overrightarrow{AC} = \left(\overrightarrow{AD} + \frac{1}{2}\overrightarrow{BC}\right) \cdot \left(\overrightarrow{AD} + \frac{1}{2}\overrightarrow{BC}\right) \\ &= \overrightarrow{AD} \cdot \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AD} \cdot \overrightarrow{BC} + \frac{1}{2}\overrightarrow{BC} \cdot \overrightarrow{AD} + \frac{1}{4}\overrightarrow{BC} \cdot \overrightarrow{BC} \\ &= |\overrightarrow{AD}|^2 + \overrightarrow{BC} \cdot \overrightarrow{AD} + \frac{1}{4}|\overrightarrow{BC}|^2 \end{aligned}$$

$$|\overrightarrow{AB}|^2 = \frac{1}{4}|\overrightarrow{BC}|^2 - \overrightarrow{BC} \cdot \overrightarrow{AD} + |\overrightarrow{AD}|^2$$

$$|\overrightarrow{AC}|^2 = |\overrightarrow{AD}|^2 + \overrightarrow{AD} \cdot \overrightarrow{BC} + \frac{1}{4}|\overrightarrow{BC}|^2$$

$$\therefore |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2|\overrightarrow{AD}|^2 + \frac{1}{2}|\overrightarrow{BC}|^2$$