VECTORS (Q 2, PAPER 2)

2007

2 (a) $\vec{x} = -2\vec{i} + 5\vec{j}$ and $\overrightarrow{xy} = -6\vec{i} - 8\vec{j}$. Express $\vec{y}$ in terms of $\vec{i}$ and $\vec{j}$.

(b) $\vec{a} = 5\vec{i}$ and $\vec{b} = \sqrt{3}\vec{i} + 3\vec{j}$.

(i) Show that $\overrightarrow{ab}$ is not perpendicular to $\vec{b}$.

(ii) Find the value of the real number k, given that $\vec{c} = k\vec{b}$ and $\overrightarrow{ac} \perp \vec{b}$.

(c) $\vec{p} = 3\vec{i} + 4\vec{j}$ and $\vec{q} = 5\vec{i} + 12\vec{j}$.

$$\vec{r} = \frac{65t}{16}\left(\frac{\vec{p}}{|\vec{p}|} + \frac{\vec{q}}{|\vec{q}|}\right), \text{ where } t > 0.$$

(i) Express $\vec{r}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find $\vec{p}.\vec{r}$ and $\vec{q}.\vec{r}$.

(iii) Hence, show that r is on the bisector of $\angle poq$, where o is the origin.

SOLUTION

2 (a)

$\overrightarrow{ab} = \vec{b} - \vec{a}$ 1

$\vec{x} = -2\vec{i} + 5\vec{j}$

$\overrightarrow{xy} = \vec{y} - \vec{x} = -6\vec{i} - 8\vec{j} \Rightarrow \vec{y} = -6\vec{i} - 8\vec{j} + \vec{x}$

$\Rightarrow \vec{y} = -6\vec{i} - 8\vec{j} - 2\vec{i} + 5\vec{j} = -8\vec{i} - 3\vec{j}$

2 (b) (i)

$\vec{a} = 5\vec{i}$ and $\vec{b} = \sqrt{3}\vec{i} + 3\vec{j}$.

If two vectors are perpendicular their dot product is zero.

$\overrightarrow{ab}.\vec{b} = (\vec{b} - \vec{a}).\vec{b} = ((\sqrt{3} - 5)\vec{i} + 3\vec{j}).(\sqrt{3}\vec{i} + 3\vec{j})$

$\sqrt{3}(\sqrt{3} - 5) + 9 = 3 - 5\sqrt{3} + 9 = 12 - 5\sqrt{3} \neq 0$. Therefore, they are not perpendicular.

2 (b) (ii)

$\vec{c} = k\vec{b} = k(\sqrt{3}\vec{i} + 3\vec{j}) = \sqrt{3}k\vec{i} + 3k\vec{j}$

$\overrightarrow{ac} = \vec{c} - \vec{a} = \sqrt{3}k\vec{i} + 3k\vec{j} - 5\vec{i} = (\sqrt{3}k - 5)\vec{i} + 3k\vec{j}$

If $\overrightarrow{ac} \perp \vec{b} \Rightarrow \overrightarrow{ac}.\vec{b} = 0.$

$\therefore \overrightarrow{ac}.\vec{b} = ((\sqrt{3}k - 5)\vec{i} + 3k\vec{j}).(\sqrt{3}\vec{i} + 3\vec{j}) = 0$

$\Rightarrow \sqrt{3}(\sqrt{3}k - 5) + 9k = 0 \Rightarrow 3k - 5\sqrt{3} + 9k = 0$

$\Rightarrow 12k = 5\sqrt{3} \Rightarrow k = \dfrac{5\sqrt{3}}{12}$

$\boxed{\vec{i}.\vec{j} = \vec{j}.\vec{i} = 0}$ 9

$\boxed{\vec{i}.\vec{i} = \vec{j}.\vec{j} = |\vec{i}|^2 = |\vec{j}|^2 = 1}$ 10

Trick: Just multiply the $\vec{i}$ coefficients and multiply the $\vec{j}$ coefficients.

2 (c) (i)

Unit Vector: $\dfrac{\vec{r}}{|\vec{r}|} = \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ 6

$$\vec{r} = \frac{65t}{16}\left(\frac{\vec{p}}{|\vec{p}|} + \frac{\vec{q}}{|\vec{q}|}\right) = \frac{65t}{16}\left(\frac{3\vec{i} + 4\vec{j}}{\sqrt{9+16}} + \frac{5\vec{i} + 12\vec{j}}{\sqrt{25+144}}\right)$$

$$= \frac{65t}{16}\left(\frac{3\vec{i} + 4\vec{j}}{5} + \frac{5\vec{i} + 12\vec{j}}{13}\right) = \frac{65t}{16}\left(\frac{39\vec{i} + 52\vec{j} + 25\vec{i} + 60\vec{j}}{65}\right)$$

$$= \frac{65t}{16}\left(\frac{3\vec{i} + 4\vec{j}}{5} + \frac{5\vec{i} + 12\vec{j}}{13}\right) = \frac{65t}{16}\left(\frac{64\vec{i} + 112\vec{j}}{65}\right) = t(4\vec{i} + 7\vec{j})$$

2 (c) (ii)

$\vec{p}.\vec{r} = (3\vec{i} + 4\vec{j}).(4t\vec{i} + 7t\vec{j}) = 12t + 28t = 40t$

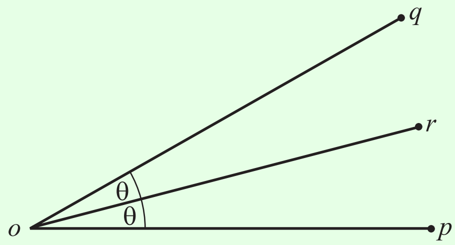
$\vec{q}.\vec{r} = (5\vec{i} + 12\vec{j}).(4t\vec{i} + 7t\vec{j}) = 20t + 84t = 104t$

2 (c) (iii)

Dot product: $\boxed{\overrightarrow{ab}.\overrightarrow{ac} = |\overrightarrow{ab}||\overrightarrow{ac}|\cos\theta}$ 8

$\vec{p}.\vec{r} = |\vec{p}||\vec{r}|\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{p}.\vec{r}}{|\vec{p}||\vec{r}|}$

$\vec{q}.\vec{r} = |\vec{q}||\vec{r}|\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{q}.\vec{r}}{|\vec{q}||\vec{r}|}$



If r bisects the angle $\Rightarrow \dfrac{\vec{p}.\vec{r}}{|\vec{p}||\vec{r}|} = \dfrac{\vec{q}.\vec{r}}{|\vec{q}||\vec{r}|}$

$$\Rightarrow \frac{\vec{p}.\vec{r}}{|\vec{p}||\vec{r}|} = \frac{40t}{5\sqrt{16t^2 + 49t^2}} = \frac{40t}{5\sqrt{65t^2}} = \frac{40t}{5t\sqrt{65}} = \frac{8}{\sqrt{65}}$$

$$\Rightarrow \frac{\vec{q}.\vec{r}}{|\vec{q}||\vec{r}|} = \frac{104t}{13t\sqrt{65}} = \frac{8}{\sqrt{65}}$$

Therefore, r is on the bisector of $\angle poq$.