1999

4 (a) Find the value of k for which

$$\sin 75^\circ - \sin 15^\circ = \frac{1}{\sqrt{k}},\ k \in \mathbf{N}_0.$$

See tables page 9.

(b) (i) Express $\sin 5x - \sin x$ as a product of sine and cosine.

(ii) Find all the solutions of the equation

$$\sin 5x - \sin x = 0$$

in the domain $0^\circ \le x \le 180^\circ$.

(c) Prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Find, in the form $p \pm \sqrt{q},\ p \in \mathbf{Z},\ q \in \mathbf{N}$,

(i) $\tan 75^\circ$

(ii) $\tan 15^\circ$.

SOLUTION

4 (a)

Change the sum into a product:

$$\sin 75^\circ - \sin 15^\circ = 2\cos\left(\tfrac{75^\circ + 15^\circ}{2}\right)\sin\left(\tfrac{75^\circ - 15^\circ}{2}\right) = 2\cos 45^\circ \sin 30^\circ$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$$ [Using the table below.]

A	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
A	0°	180°	90°	60°	45°	30°
$\cos A$	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin A$	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan A$	0	0	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

SUMS $\rightarrow$ PRODUCTS
$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\therefore k = 2$

4 (b) (i)

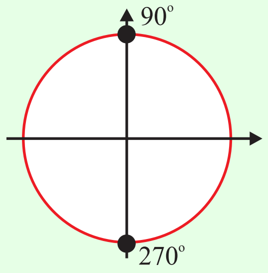
$$\sin 5x - \sin x = 2\cos(\tfrac{5x+x}{2})\sin(\tfrac{5x-x}{2}) = 2\cos 3x \sin 2x$$

Sums $\longrightarrow$ Products
$\sin A + \sin B = 2\sin(\tfrac{A+B}{2})\cos(\tfrac{A-B}{2})$
$\sin A - \sin B = 2\cos(\tfrac{A+B}{2})\sin(\tfrac{A-B}{2})$
$\cos A + \cos B = 2\cos(\tfrac{A+B}{2})\cos(\tfrac{A-B}{2})$
$\cos A - \cos B = -2\sin(\tfrac{A+B}{2})\sin(\tfrac{A-B}{2})$

4 (b) (ii)

Solve $2\cos 3x \sin 2x = 0 \Rightarrow \cos 3x \sin 2x = 0$

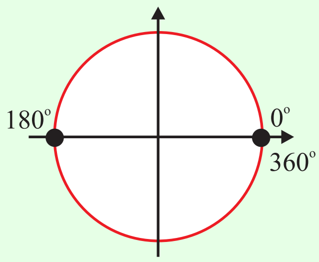
$\therefore \cos 3x = 0$



$3x \quad = 90^\circ, 450^\circ$
$\qquad = 270^\circ, 630^\circ$

$x \quad = 30^\circ, 150^\circ$
$\qquad = 90^\circ$, ~~210°~~

$\therefore \sin 2x = 0$



$2x \quad = 0^\circ, 360^\circ$
$\qquad = 180^\circ, 540^\circ$

$x \quad = 0^\circ, 180^\circ$
$\qquad = 90^\circ$, ~~270°~~

$\therefore\ x = 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ$

4 (c)

This is a trig identity.

Steps

1. You have to prove that the left-hand side (*LHS*) equals the right-hand side (*RHS*).
2. Change everything to sine and cosine.
3. Simplify each side using page 9 of the tables and good algebra.
4. For half angles, $\frac{\theta}{2}$: Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$.

LHS

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

RHS

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - (\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - (\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})} \times \frac{\cos A \cos B}{\cos A \cos B}$$

$$= \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

LHS = *RHS*

4 (c) (i)

$\tan 75^{o} = \tan(45^{o} + 30^{o})$ [Written as a sum of famous angles whose values are given in the tables on page 9.]

$$= \frac{\tan 45^{o} + \tan 30^{o}}{1 - \tan 45^{o} \tan 30^{o}} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)(\frac{1}{\sqrt{3}})}$$

A	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
A	0^{o}	180^{o}	90^{o}	60^{o}	45^{o}	30^{o}
$\cos A$	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin A$	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan A$	0	0	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

Multiply above and below by $\sqrt{3}$.

$$\therefore \tan 75^{o} = \frac{(1 + \frac{1}{\sqrt{3}})}{(1 - \frac{1}{\sqrt{3}})} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Rationalise the denominator by multiplying above and below by the conjugate.

$$\therefore \tan 75^{o} = \frac{(\sqrt{3} + 1)}{(\sqrt{3} - 1)} \times \frac{(\sqrt{3} + 1)}{(\sqrt{3} + 1)} = \frac{3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

4 (c) (ii)

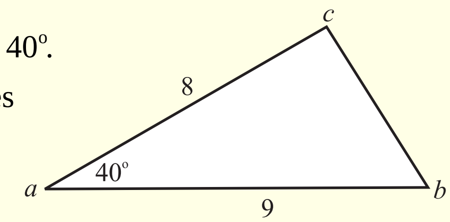
$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ [Change the sign in the original formula]

$$\tan 15^{o} = \tan(45^{o} - 30^{o}) = \frac{\tan 45^{o} - \tan 30^{o}}{1 + \tan 45^{o} \tan 30^{o}} = \frac{(1 - \frac{1}{\sqrt{3}})}{(1 + \frac{1}{\sqrt{3}})} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{(\sqrt{3} - 1)}{(\sqrt{3} + 1)} \times \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)} = \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

5 (a) In the triangle abc, $|ab| = 9$, $|ac| = 8$ and $|\angle cab| = 40^\circ$.

Find the area of triangle abc, correct to two places of decimals.



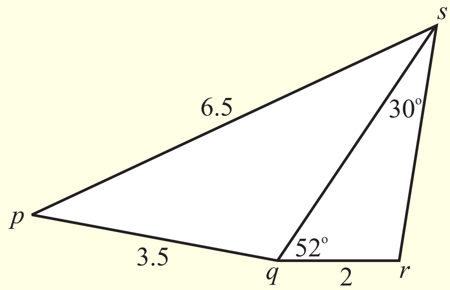
(b) In the triangles pqs and qrs, $|pq| = 3\cdot5$, $|qr| = 2$, $|ps| = 6\cdot5$, $|\angle qsr| = 30^\circ$ and

$|\angle sqr| = 52^\circ$.

Calculate

(i) $|qs|$, correct to two places of decimals

(ii) $|\angle pqs|$, correct to the nearest degree.



(c) Express $\sin(135^\circ - A)$ in terms of $\sin A$ and $\cos A$.

Express $\sin(135^\circ - A)\cos(135^\circ + A)$ in the form $k(1 + \sin pA)$, where $k,\ p \in \mathbf{R}$.

Find the values of A for which

$$\sin(135^\circ - A)\cos(135^\circ + A) = -\tfrac{3}{4}$$

where $0^\circ \le A \le 180^\circ$.

Solution

5 (a)

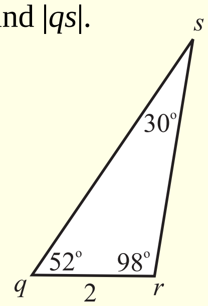
$A = \frac{1}{2}(8)(9)\sin 40^\circ = 23.14$ square units

$A = \frac{1}{2}ab\sin C$ **5**

5 (b) (i)

Start with triangle qrs as it has the most information. It is easy to find the third angle.

Use the Sine Rule to find $|qs|$.



The Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ **18**

You use the Sine Rule when you are given:

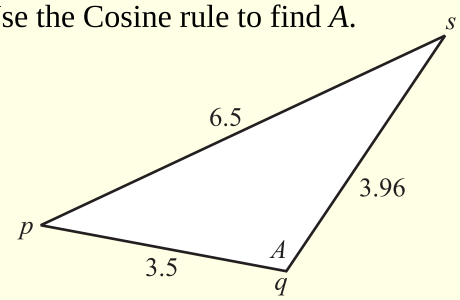
1. 2 sides and 1 non-included angle,

2. 2 angles and 1 side.

$$\therefore \frac{|qs|}{\sin 98^\circ} = \frac{2}{\sin 30^\circ} \Rightarrow |qs| = \frac{2\sin 98^\circ}{\sin 30^\circ} = 3.96 \text{ square units}$$

5 (b) (ii)

Move into triangle *pqs*. Call the angle you need to find, *A*. Use the Cosine rule to find *A*.



THE COSINE RULE

$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

You use the Cosine rule when you are given:

1. 2 sides and 1 included angle,

2. 3 sides.

$\therefore 6.5^2 = 3.5^2 + 3.96^2 - 2(3.5)(3.96)\cos A$

$\Rightarrow A = \cos^{-1}\left(\frac{6.5^2 - 3.5^2 + 3.96^2}{-2(3.5)(3.96)}\right) = 121^\circ$

5 (c)

Use formula 10.

$\sin(A - B) = \sin A\cos B - \cos A\sin B$ **10**

$\sin(135^\circ - A) = \sin 135^\circ\cos A - \cos 135^\circ\sin A$

[Using ASTC, $\sin 135^\circ = \sin 45^\circ$ and $\cos 135^\circ = -\cos 45^\circ$]

$\therefore \sin(135^\circ - A) = \sin 45^\circ\cos A + \cos 45^\circ\sin A = \frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A$ [Using the table on p.9]

$\therefore \sin(135^\circ - A) = \frac{1}{\sqrt{2}}(\cos A + \sin A)$

$\cos(135^\circ + A) = \cos 135^\circ\cos A - \sin 135^\circ\sin A$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ **11**

$= -\cos 45^\circ\cos A - \sin 45^\circ\sin A = -\frac{1}{\sqrt{2}}\cos A - \frac{1}{\sqrt{2}}\sin A = -\frac{1}{\sqrt{2}}(\cos A + \sin A)$

$\therefore \sin(135^\circ - A)\cos(135^\circ + A) = [\frac{1}{\sqrt{2}}(\cos A + \sin A)][-\frac{1}{\sqrt{2}}(\cos A + \sin A)]$

$= -\frac{1}{2}(\cos A + \sin A)^2 = -\frac{1}{2}(\cos^2 A + 2\sin A\cos A + \sin^2 A)$

$\cos^2 A + \sin^2 A = 1$ **8**

$= -\frac{1}{2}(1 + \sin 2A)$ [Using formulae **8** and **13**.]

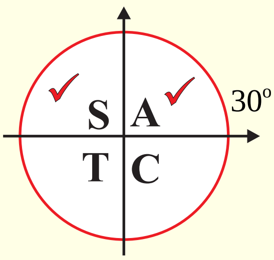
$\sin 2A = 2\sin A\cos A$ **13**

$\sin(135^\circ - A)\cos(135^\circ + A) = -\frac{3}{4} \Rightarrow -\frac{1}{2}(1 + \sin 2A) = -\frac{3}{4}$

$\Rightarrow 1 + \sin 2A = \frac{3}{2} \Rightarrow \sin 2A = \frac{1}{2}$

$\Rightarrow 2A = \sin^{-1}(\frac{1}{2}) = 30^\circ$

2*A* = 30°, 390° (First quadrant)
= 150° (Second quadrant)



A = 15°, ~~195°~~
= 75°

$\therefore A = 15^\circ, 75^\circ$