TRIGONOMETRY (Q 4 & 5, PAPER 2)

1998

4 (a) Find the values of θ for which $\cos\theta = \frac{\sqrt{3}}{2}$, where $0^\circ \le \theta \le 360^\circ$.

(b) Find the two solutions of the equation

$$4\sin^2 x - 3\cos x - 3 = 0,$$

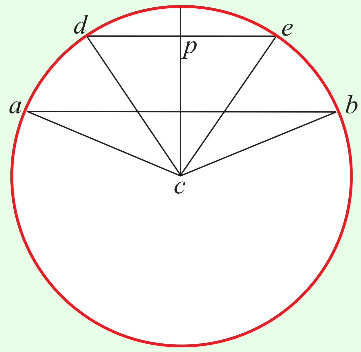
where $0^\circ \le x \le 180^\circ$.
Give your answers correct to the nearest degree.

(c) $[ab]$ and $[de]$ are two parallel chords of a circle with centre c and radius length r.

$cp \perp de, |\angle acb| = 4\beta$ and $|\angle dce| = 2\beta$, where β is in radian measure, $\beta \ne 0$.

(i) If the area of the triangle acb equals the area of triangle dce, show that $\beta = \frac{\pi}{6}$.

(ii) Calculate the value of r if $|ab|^2 + |de|^2 = 24$ and give your answer as a surd.

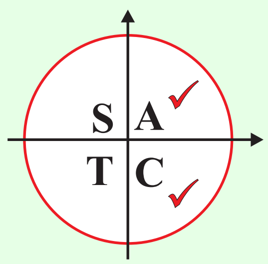


SOLUTION

4 (a)

$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}(\frac{\sqrt{3}}{2}) = 30^\circ$

θ = 30° (First quadrant)
= 330° (Second quadrant)



4 (b)

$4\sin^2 x - 3\cos x - 3 = 0 \Rightarrow 4(1-\cos^2 x) - 3\cos x - 3 = 0$ [Using formula **8**.]

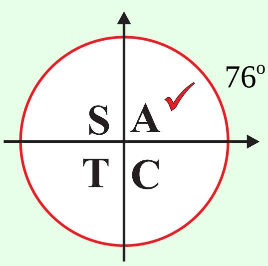
$\Rightarrow 4 - 4\cos^2 x - 3\cos x - 3 = 0$

$\Rightarrow 4\cos^2 x + 3\cos x - 1 = 0$

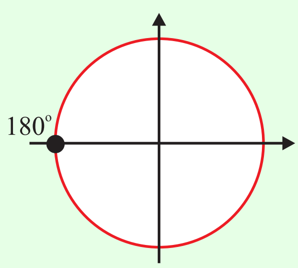
$\Rightarrow (4\cos x - 1)(\cos x + 1) = 0$

$\cos^2 A + \sin^2 A = 1$ **8**

$\therefore \cos x = \frac{1}{4} \Rightarrow x = \cos^{-1}(\frac{1}{4}) = 76^\circ$



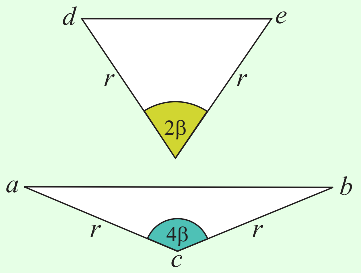
$\therefore \cos x = -1$

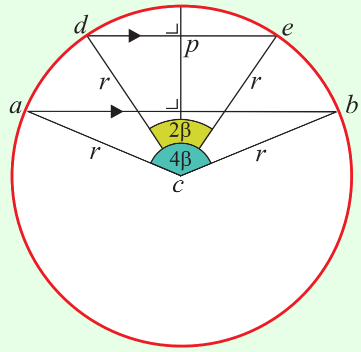


$\therefore x = 76^\circ, 180^\circ$

4 (c) (i)

Area of triangle *acb* = Area of triangle *dce*





$A = \frac{1}{2}ab\sin C$ **5**

Area of triangle *acb* $= \frac{1}{2}(r)(r)\sin 4\beta = \frac{1}{2}r^2 \sin 4\beta$

Area of triangle *dce* $= \frac{1}{2}(r)(r)\sin 2\beta = \frac{1}{2}r^2 \sin 2\beta$

$\therefore \frac{1}{2}r^2 \sin 4\beta = \frac{1}{2}r^2 \sin 2\beta \Rightarrow 2\sin 2\beta \cos 2\beta = \sin 2\beta$

$\Rightarrow 2\cos 2\beta = 1 \Rightarrow \cos 2\beta = \frac{1}{2}$

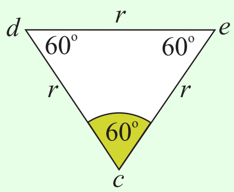
$\Rightarrow 2\beta = \cos^{-1}(\frac{1}{2}) = 60^\circ = \frac{\pi}{3}$

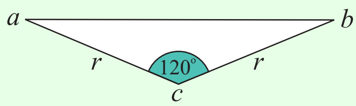
$\therefore \beta = \frac{\pi}{6}$

$\sin 2A = 2\sin A\cos A$ **13**

$\sin 4\beta = \sin 2(2\beta) = 2\sin 2\beta \cos 2\beta$

4 (c) (ii)





$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

From triangle *dce*: $|de| = r$

From triangle *acb*: $|ab|^2 = r^2 + r^2 - 2r^2\cos 120^\circ = 2r^2 + 2r^2\cos 60^\circ$

$\Rightarrow |ab|^2 = 2r^2 + 2r^2(\frac{1}{2}) = 3r^2$

Now $|ab|^2 + |de|^2 = 24 \Rightarrow r^2 + 3r^2 = 24 \Rightarrow 4r^2 = 24$

$\therefore r^2 = 6 \Rightarrow r = \sqrt{6}$

5 (a) Express $\sin A$ in terms of t if

$\tan A = \frac{t}{2}$, where $t > 0$ and $0^\circ < A < 90^\circ$.

(b) If $\tan A = \frac{1}{2}$, find $\tan 2A$ without evaluating A, where A is an acute angle.

Express $\tan B$ in the form $\frac{a}{b}$, where $a, b \in \mathbf{N}_0$, given that

$\tan(2A + B) = \frac{63}{16}$.

(c) Express $\sin 2A + \sin 2B$ as a product in sine and cosine.

If $A + B + C = 180^\circ$, show that

$\sin(A + B) = \sin C$.

Hence, show that

$\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$.

Note: $\cos(A + B) = -\cos C$.

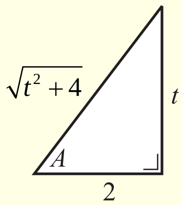
SOLUTION

5 (a)

$\sin A = \frac{y}{r} = \frac{\text{Opposite}}{\text{Hypotenuse}}$ 2

$\tan A = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ 3

$x^2 + y^2 = r^2$ 4



Using Pythagoras: $t^2 + 4 = r^2 \Rightarrow r = \sqrt{t^2 + 4}$

$$\therefore \sin A = \frac{t}{\sqrt{t^2 + 4}}$$

5 (b)

$$\tan 2A = \frac{2(\frac{1}{2})}{1 - (\frac{1}{2})^2} = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$ 17

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ 15

$$\tan(2A + B) = \frac{63}{16} \Rightarrow \frac{\tan 2A + \tan B}{1 - \tan 2A \tan B} = \frac{\frac{4}{3} + \tan B}{1 - \frac{4}{3}\tan B} = \frac{63}{16}$$

$$\Rightarrow \frac{4}{3} + \tan B = \frac{63}{16}(1 - \frac{4}{3}\tan B) \Rightarrow \frac{4}{3} + \tan B = \frac{63}{16} - \frac{21}{4}\tan B$$

$$\Rightarrow \tan B + \frac{21}{4}\tan B = \frac{63}{16} - \frac{4}{3}$$

$$\Rightarrow \frac{25}{4}\tan B = \frac{125}{48} \Rightarrow \tan B = \frac{125}{48} \times \frac{4}{25} = \frac{5}{12}$$

5 (c)

$\sin 2A + \sin 2B = 2\sin(\frac{2A+2B}{2})\cos(\frac{2A-2B}{2})$

$= 2\sin(A+B)\cos(A-B)$

$A+B+C = 180^o \Rightarrow A+B = 180^o - C$

$\Rightarrow \sin(A+B) = \sin(180^o - C)$

Sums $\to$ Products
$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$
$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$
$\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$
$\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ 10

A	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$
A	0^o	180^o	90^o	60^o	45^o	30^o
$\cos A$	1	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin A$	0	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan A$	0	0	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

$\sin(180^o - C) = \sin 180^o \cos C - \cos 180^o \sin C$

$\therefore \sin(180^o - C) = (0)\cos C - (-1)\sin C = \sin C$

$\therefore \sin(A+B) = \sin C$

$(\sin 2A + \sin 2B) - \sin 2C = 2\sin(A+B)\cos(A-B) - 2\sin C\cos C$

$\sin 2A = 2\sin A\cos A$ 13

You are told that $\cos(A+B) = -\cos C$

$\Rightarrow \sin 2A + \sin 2B - \sin 2C = 2\sin C\cos(A-B) - 2\sin C[-\cos(A+B)]$

$= 2\sin C\cos(A-B) + 2\sin C\cos(A+B)$

$= 2\sin C[\cos(A-B) + \cos(A+B)]$ [Change a sum to a product]

$= 2\sin C[2\cos(\frac{A-B+A+B}{2})\cos(\frac{A-B-A-B}{2})]$

$= 2\sin C[2\cos A\cos(-B)]$

$\cos(-A) = \cos A$

$= 4\cos A\cos B\sin C$