Trigonometry (Q 4 & 5, Paper 2)

1997

4 (a) Find the value for θ for which

$\tan\theta = -\frac{1}{\sqrt{3}}$

where $0^\circ \le \theta \le 180^\circ$.

(b) In a triangle pqr, $|\angle pqr| = 30^\circ, |qr| = 15$ and $|rp| = 5\sqrt{3}$.

Find two values for $|\angle qpr|$ and sketch the two resulting triangles.
Calculate the ratio of the areas of the two triangles.

(c) Show, using the formula for sin $(A + B)$, that
sin $2A$ = 2sin Acos A.

Using the tables on page 9, or otherwise, show that

$\sin 3A = 3\sin A - 4\sin^3 A.$

Use the result for sin $3A$, or otherwise, to show that

$\sin 3(A - \frac{\pi}{2}) = 4\cos^3 A - 3\cos A.$

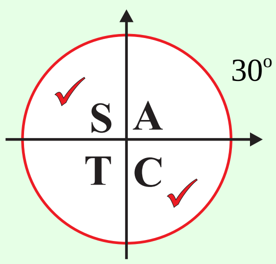
Solution

4 (a)

$\tan\theta = -\frac{1}{\sqrt{3}}$

Find the basic angle in the first quadrant.

$\theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = 30^\circ$

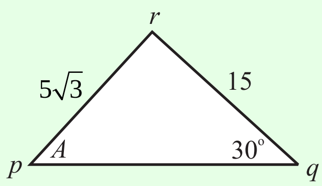


θ = 150° (Second quadrant)
= ~~330°~~ (Fourth quadrant) [Outside the range]

$\therefore \theta = 150^\circ$

4 (b)

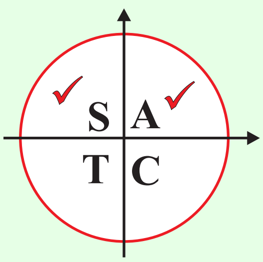
Call the angle A. Use the Sine rule.



$$\frac{\sin A}{15} = \frac{\sin 30^\circ}{5\sqrt{3}} \Rightarrow \sin A = \frac{15\sin 30^\circ}{5\sqrt{3}} = \frac{15(\frac{1}{2})}{5\sqrt{3}} = \frac{\sqrt{3}}{2}$$

Basic angle: $A = \sin^{-1}(\frac{\sqrt{3}}{2}) = 60^\circ$

A = 60° (First quadrant)
= 120° (second quadrant)

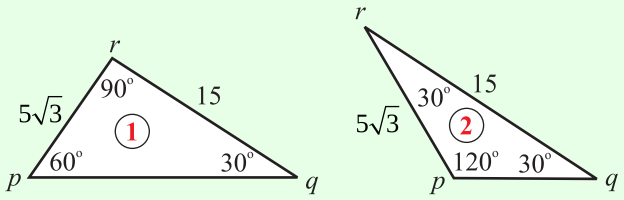


The Sine Rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \ldots\ldots \ \textbf{18}$$

You use the Sine Rule when you are given:

1. 2 sides and 1 non-included angle,
2. 2 angles and 1 side.



Area of triangle 1: $A_1 = \frac{1}{2}(5\sqrt{3})(15)\sin 90^\circ = \dfrac{75\sqrt{3}}{2}$ square units $\boxed{A = \frac{1}{2}ab\sin C}$ **5**

Area of triangle 2: $A_2 = \frac{1}{2}(5\sqrt{3})(15)\sin 30^\circ = \frac{1}{2}(5\sqrt{3})(15)(\frac{1}{2}) = \dfrac{75\sqrt{3}}{4}$ square units

$$A_1 : A_2 = \frac{75\sqrt{3}}{2} : \frac{75\sqrt{3}}{4} = 2:1$$

4 (c) $\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$ **9**

Replace *B* with *A* in the formula for sin (*A* + *B*).

$\sin(A+A) = \sin A\cos A + \cos A\sin A \Rightarrow \sin 2A = 2\sin A\cos A$

Required to prove: $\sin 3A = 3\sin A - 4\sin^3 A$

LHS

$\sin 3A = \sin(2A + A)$

$= \sin 2A\cos A + \cos 2A\sin A$

$= (2\sin A\cos A)\cos A + (\cos^2 A - \sin^2 A)\sin A$

$= 2\sin A\cos^2 A + \cos^2 A\sin A - \sin^3 A$

$= 2\sin A(1 - \sin^2 A) + (1 - \sin^2 A)\sin A - \sin^3 A$

$= 2\sin A - 2\sin^3 A + \sin A - \sin^3 A - \sin^3 A$

$= 3\sin A - 4\sin^3 A$

= *RHS*

$\boxed{\sin 2A = 2\sin A\cos A}$ **13**

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$ **14**

$\boxed{\cos^2 A + \sin^2 A = 1}$ **8**

$\Rightarrow \cos^2 A = 1 - \sin^2 A$

Replace *A* by $A - \frac{\pi}{2}$.

$\therefore \sin 3(A - \frac{\pi}{2}) = 3\sin(A - \frac{\pi}{2}) - 4\sin^3(A - \frac{\pi}{2}) = 3\sin(A - \frac{\pi}{2}) - 4[\sin(A - \frac{\pi}{2})]^3$

$\sin(A - \frac{\pi}{2}) = \sin(A - 90^\circ) = \sin A\cos 90^\circ - \cos A\sin 90^\circ = \sin A \times (0) - \cos A \times (1) = -\cos A$

$\therefore \sin 3(A - \frac{\pi}{2}) = 3(-\cos A) - 4[(-\cos A)]^3 = 4\cos^3 A - 3\cos A$

5 (a) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$.

(b) (i) Express sin 5x + sin 3x as a product of sine and cosine.

(ii) Find all the solutions of the equation
sin 5x + sin 3x = 0
in the domain $0^\circ \le \theta \le 180^\circ$.

(c) A triangle has sides of length a, b, and c with A being the angle opposite the side of length a.
Derive a formula for a^2 in terms of b, c and A.
When $90^\circ < A < 180^\circ$ prove that
$a^2 > b^2 + c^2$.

Solution

5 (a)

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \times \frac{3\cancel{x}}{\cancel{x}} = 3$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$ 20

5 (b) (i)

$$\sin 5x + \sin 3x = 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2\sin 4x \cos x$$

Sums → Products

$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$

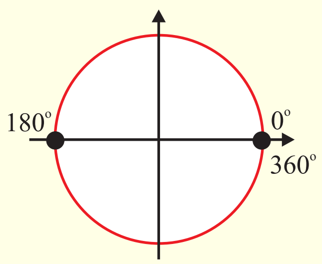
$\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$

5 (b) (ii)

$$\sin 5x + \sin 3x = 0 \Rightarrow 2\sin 4x \cos x = 0$$

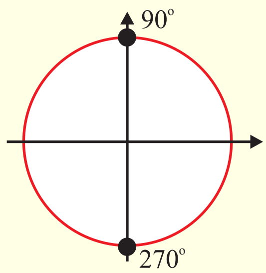
$\therefore \sin 4x = 0$



$4x$ = 0°, 360°, 720°
= 180°, 540°

x = 0°, 90°, 180°
= 45°, 135°

$\therefore \cos x = 0$



x = 90°

x = 0°, 45°, 90°, 135°, 180°

5 (c)

You are being asked to prove the Cosine Rule.

PROOF OF THE COSINE RULE

Applying Pythagoras to right-angled triangle **1**:

$a^2 = (c-x)^2 + h^2$

$\Rightarrow a^2 = (h^2 + x^2) + c^2 - 2cx$

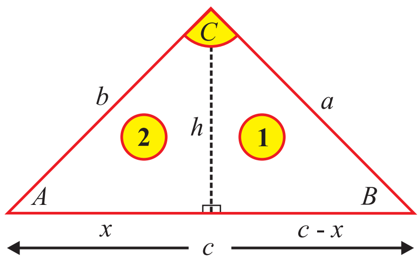
Applying Pythagoras to right-angled triangle **2**:

$b^2 = h^2 + x^2$

$\Rightarrow a^2 = b^2 + c^2 - 2cx$

Also $\cos A = \frac{x}{b} \Rightarrow x = b\cos A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$



$a^2 = b^2 + c^2 - 2bc\cos A$

As $90^\circ < A < 180^\circ \Rightarrow \cos A$ is a negative number.

$\therefore a^2 = b^2 + c^2 + 2bc\cos A$

As $2bc\cos A > 0 \Rightarrow a^2 > b^2 + c^2$