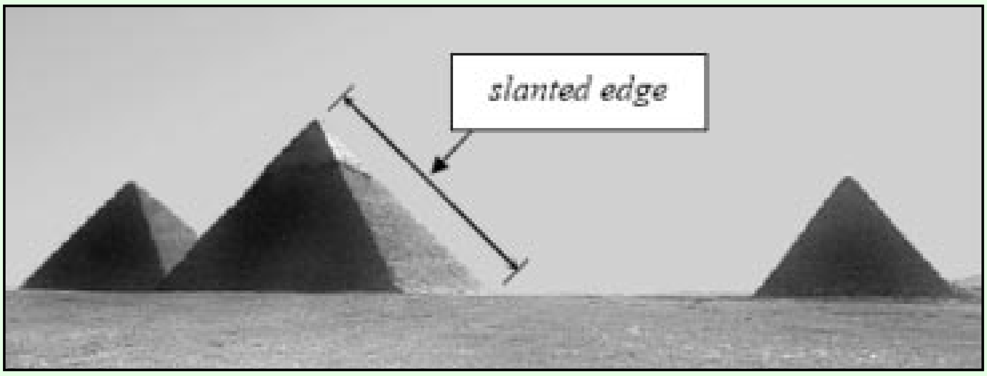
TRIGONOMETRY (Q 4 & 5, PAPER 2)

LESSON NO. 8: SOLVING TRIANGLES

2006

5 (b) The great pyramid at Giza in Eqypt has a square base and four triangular faces. The base of the pyramid is of side 230 metres and the pyramid is 146 metres high. The top of the pyramid is directly above the centre of the base.



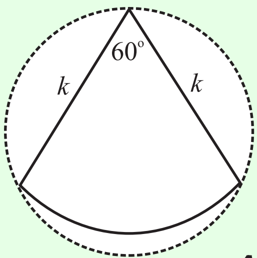
(i) Calculate the length of one of the slanted edges, correct to the nearest metre.

(ii) Calculate, correct to two significant figures, the total area of the four triangular faces of the pyramid (assuming they are smooth flat surfaces).

4 (c) The diagram shows a sector (solid line) circumscribed by a circle (dashed line).

(i) Find the radius of the circle in terms of k.

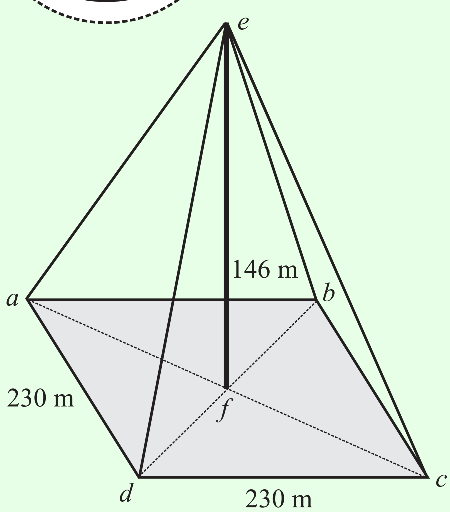
(ii) Show that the circle encloses an area which is double that of the sector.



SOLUTION

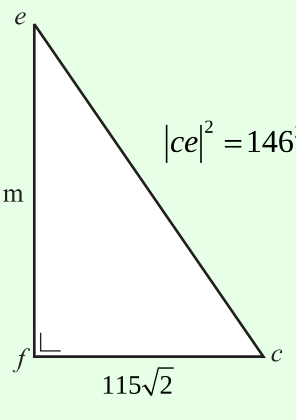
5 (b) (i)

Using Pythagoras calculate the length of the diagonal of the square base.

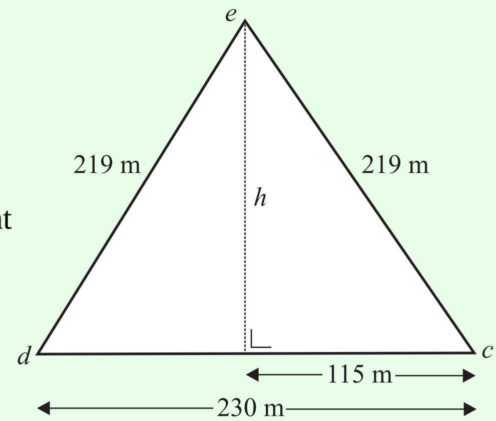


$$|bd|^2 = 230^2 + 230^2 \Rightarrow |bd| = 230\sqrt{2}$$

Pick out Δefc.



$$|ce|^2 = 146^2 + (115\sqrt{2})^2 \Rightarrow |ce| = 219 \text{ m}$$



5 (b) (ii)

Pick out Δedc. Using Pythagoras, calculate the height h of the triangular face.

$\therefore h^2 + 115^2 = 219^2 \Rightarrow h = 186 \text{ m}$

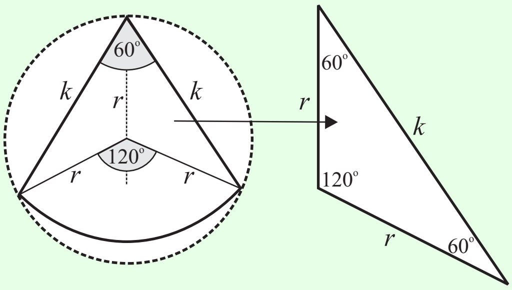
Area of a triangle: $A = \frac{1}{2} \times \text{Base} \times \text{Height}$

Area of four triangular faces:

$A = 4 \times \frac{1}{2} \times 230 \times 186 = 85{,}560 \text{ m}^2 \approx 86{,}000 \text{ m}^2$ [Correct to 2 significant figures.]

4 (c) (i)

From the Junior Cert., you might remember that the angle at the centre of a circle is twice the angle on the circle standing on the same arc.



$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

Using the Cosine rule: $k^2 = r^2 + r^2 - 2(r)(r)\cos 120^\circ \Rightarrow k^2 = 2r^2 - 2r^2\cos 120^\circ$

$\Rightarrow k^2 = 2r^2(1 - \cos 120^\circ) \Rightarrow k^2 = 2r^2(1 - \cos 120^\circ)$

$\Rightarrow k^2 = 2r^2(1 + \cos 60^\circ) \Rightarrow k^2 = 2r^2(1 + \frac{1}{2}) \Rightarrow k^2 = 2r^2(\frac{3}{2})$

$\Rightarrow k^2 = 3r^2 \Rightarrow r = \frac{k}{\sqrt{3}}$

4 (c) (ii)

Area of circle: $A_1 = \pi r^2 = \pi\left(\frac{k^2}{3}\right) = \frac{1}{3}\pi k^2$

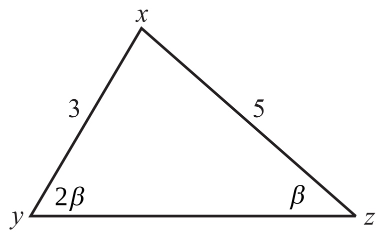
Area of sector: $A_2 = \frac{1}{2}k^2(\frac{\pi}{3}) = \frac{1}{6}\pi k^2$ $A = \frac{1}{2}r^2\theta$ **7**

$\therefore A_1 = 2A_2$

2005

5 (b) In the triangle xyz, $|\angle xyz| = 2\beta$ and $|\angle xzy| = \beta$.

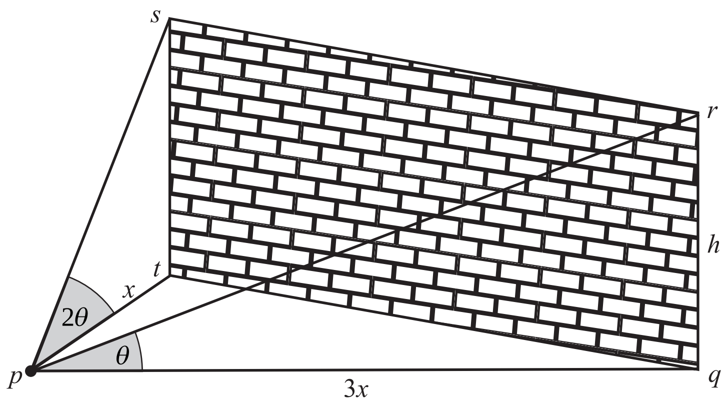
$|xy| = 3$ and $|xz| = 5$.



(i) Use this information to express $\sin 2\beta$ in the form $\frac{a}{b}\sin\beta$, where $a, b \in \mathbf{N}$.

(ii) Hence express $\tan\beta$ in the form $\frac{\sqrt{c}}{d}$, where $c, d \in \mathbf{N}$.

5 (c) *qrst* is a vertical rectangular wall of height *h* on level ground. *p* is a point on the ground in front of the wall. The angle of elevation of *r* from *p* is θ and the angle of elevation of *s* from *p* is 2θ. $|pq| = 3|pt|$. Find θ.

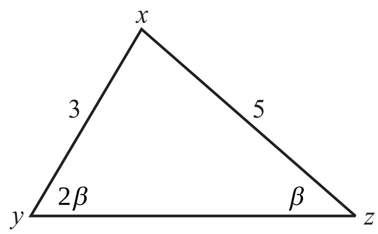


Solution

5 (b) (i)

Use the Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ **18**

$\therefore \frac{\sin 2\beta}{5} = \frac{\sin\beta}{3} \Rightarrow \sin 2\beta = \frac{5}{3}\sin\beta$



5 (b) (ii)

$\sin 2A = 2\sin A\cos A$ **13**

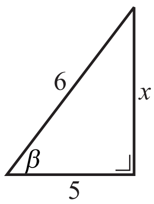
$\sin 2\beta = \frac{5}{3}\sin\beta \Rightarrow 2\sin\beta\cos\beta = \frac{5}{3}\sin\beta$

$\Rightarrow 2\cos\beta = \frac{5}{3} \Rightarrow \cos\beta = \frac{5}{6}$

$x^2 + 5^2 = 6^2 \Rightarrow x^2 = 36 - 25 = 11$

$\Rightarrow x = \sqrt{11}$

$\therefore \tan\beta = \frac{\sqrt{11}}{5}$



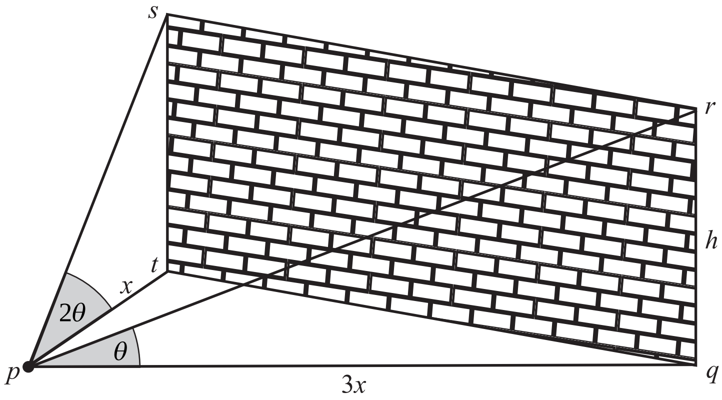
$\cos\theta = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ **1**

$\tan\theta = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ **3**

$x^2 + y^2 = r^2$ **4**

Cont....

5 (c)



Consider Δprq: $\tan\theta = \frac{h}{3x} \Rightarrow h = 3x\tan\theta$....(**1**)

Consider Δspt: $\tan 2\theta = \frac{h}{x} \Rightarrow h = x\tan 2\theta$....(**2**)

Equate (**1**) and (**2**): $\therefore 3x\tan\theta = x\tan 2\theta$

$$\Rightarrow 3\tan\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Rightarrow 3 - 3\tan^2\theta = 2$$

$$\Rightarrow 1 = 3\tan^2\theta \Rightarrow \tan\theta = \tfrac{1}{\sqrt{3}} \Rightarrow \theta = \tfrac{\pi}{6}$$

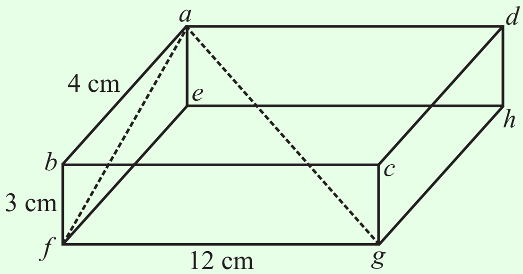
$\tan\theta = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ **3**

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ **17**

2004

5 (c) The diagram shows a rectangular box. Rectangle *abcd* is the top of the box and rectangle *efgh* is the base of the box.

$|ab| = 4$ cm, $|bf| = 3$ cm and $|fg| = 12$ cm.



(i) Find $|af|$.

(ii) Find $|ag|$.

(iii) Find the measure of the acute angle between [*ag*] and [*df*]. Give your answer correct to the nearest degree.

SOLUTION

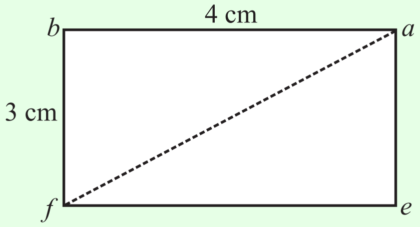
5 (c)

> **STEPS**
> **1.** Identify all right-angled triangles and non right-angled triangles and mark all angles and sides and label all vertices.
> **2.** Separate out the triangles.

5 (c) (i)

Pick out rectangle *bfea*.

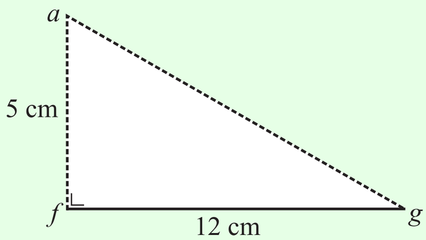
$|af|^2 = 3^2 + 4^2 = 25 \Rightarrow |af| = 5$ cm



5 (c) (ii)

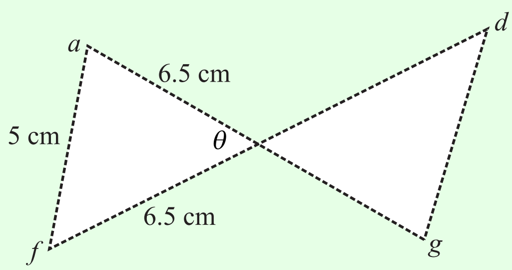
Pick out Δafg.

$|ag|^2 = 12^2 + 5^2 = 169 \Rightarrow |ag| = 13$ cm



5 (c) (iii)

The diagonals bisect each other. Now, use the cosine rule to find θ.



$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

$5^2 = 6\cdot5^2 + 6\cdot5^2 - 2(6\cdot5)(6\cdot5)\cos\theta$

$-59\cdot5 = -84\cdot5\cos\theta \Rightarrow \cos\theta = \dfrac{59\cdot5}{84\cdot5}$

$\Rightarrow \theta = 45^\circ$

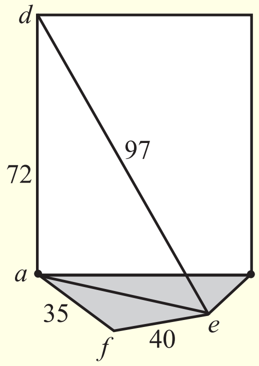
2003

5 (b) a, f and e are points on horizontal ground. d is a point on a vertical wall directly above a.

$|ad| = 72$ m, $|de| = 97$ m, $|af| = 35$ m and $|fe| = 40$ m.

(i) Calculate $|ae|$.

(ii) Hence, calculate $|\angle afe|$.



Solution

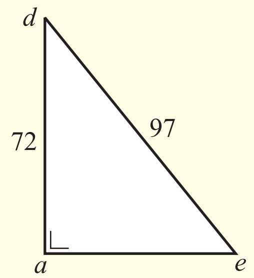
5 (b) (i)

Steps
1. Identify all right-angled triangles and non right-angled triangles and mark all angles and sides and label all vertices.
2. Separate out the triangles.

Pick out the right-angled triangle Δdae.

Using Pythagoras $72^2 + |ae|^2 = 97^2 \Rightarrow |ae|^2 = 9409 - 5184 = 4225$

$\Rightarrow |ae| = \sqrt{4225} = 65$



5 (b) (ii)

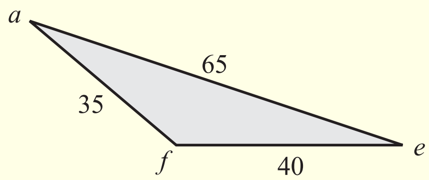
Pick out Δafe. Use the Cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

$65^2 = 35^2 + 40^2 - 2(35)(40)\cos|\angle afe|$

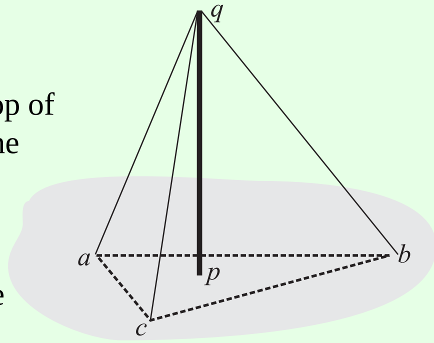
$1400 = -2800\cos|\angle afe| \Rightarrow \cos|\angle afe| = -\frac{1}{2}$

$\Rightarrow |\angle afe| = 120^\circ$



2002

5 (c) A vertical radio mast $[pq]$ stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q, to the points a, b and c on the ground. The foot of the mast, p, lies inside the triangle abc.
Each cable is 52 m long and the mast is 48 m high.



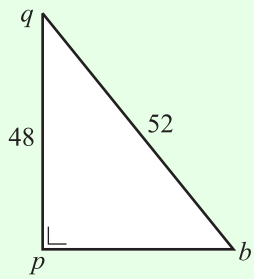
(i) Find the (common) distance from p to each of the points a, b and c.

(ii) Given that $|ac| = 38$ m and $|ab| = 34$ m, find $|bc|$ correct to one decimal place.

Solution

5 (c) (i)

Pick out the right-angled triangle Δqpb.
Using Pythagoras:

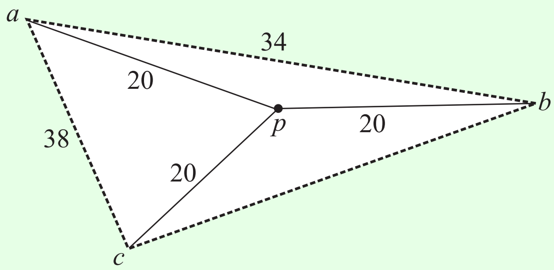


$$48^2 + |pb|^2 = 52^2 \Rightarrow |pb|^2 = 400 \Rightarrow |pb| = 20 \text{ m}$$

This is the common distance.

5 (c) (ii)

This requires the Cosine rule. You need to find angles in the first two triangles before you can solve the third triangle.



$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

Consider Δapc: $38^2 = 20^2 + 20^2 - 2(20)(20)\cos\angle apc \Rightarrow \cos\angle apc = 143 \cdot 6^{\circ}$

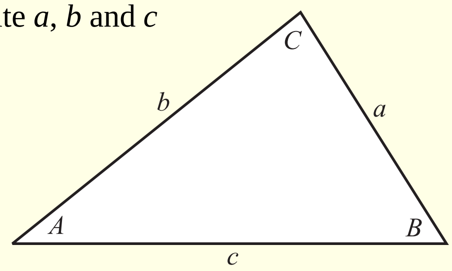
Consider Δapb: $34^2 = 20^2 + 20^2 - 2(20)(20)\cos\angle apb \Rightarrow \cos\angle apb = 116 \cdot 4^{\circ}$

Consider Δcpb: $\angle cpb = 360^{\circ} - 143 \cdot 6^{\circ} - 116 \cdot 4^{\circ} = 100^{\circ}$

$$\therefore |bc|^2 = 20^2 + 20^2 - 2(20)(20)\cos 100^{\circ} \Rightarrow |bc| = 30 \cdot 6 \text{ m}$$

2001

4 (c) A triangle has sides a, b and c. The angles opposite a, b and c are A, B and C, respectively.



(i) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

(ii) Show that $c(b\cos A - a\cos B) = b^2 - a^2$.

5 (b) xyz is a triangle where $|xy| = 8$ cm and $|yz| = 6$ cm. Given that the area of triangle xyz is 12 cm^2, find

(i) the two possible values of $|\angle xyz|$

(ii) the two possible values of $|xz|$, correct to one decimal place.

Solution

4 (c) (i)

Proof of the Cosine Rule.

Proof of the Cosine Rule

Applying Pythagoras to right-angled triangle **1**:

$a^2 = (c-x)^2 + h^2$

$\Rightarrow a^2 = (h^2 + x^2) + c^2 - 2cx$

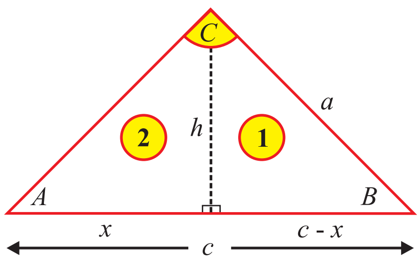
Applying Pythagoras to right-angled triangle **2**:

$b^2 = h^2 + x^2$

$\Rightarrow a^2 = b^2 + c^2 - 2cx$

Also $\cos A = \frac{x}{b} \Rightarrow x = b\cos A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$



4 (c) (ii)

Look at the angles. They are A and B. There is a minus between them. Write down the A version of the cosine rule and the B version of the cosine rule and subtract them.

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

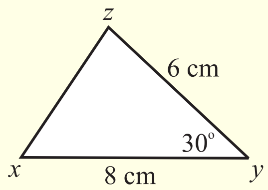
$a^2 - b^2 = b^2 - a^2 - 2bc\cos A + 2ac\cos B$

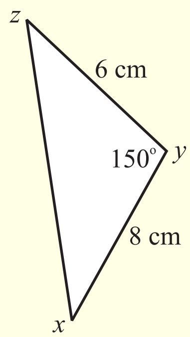
$\Rightarrow 2c(b\cos A - a\cos B) = 2b^2 - 2a^2$

$\Rightarrow c(b\cos A - a\cos B) = b^2 - a^2$

Cont....

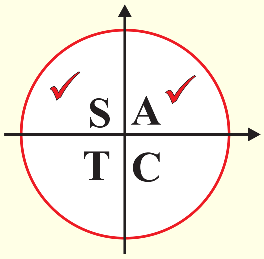
5 (b) (i) $A=\frac{1}{2}ab\sin C$ **5**





$12=\frac{1}{2}(8)(6)\sin|\angle xyz| \Rightarrow \sin|\angle xyz|=\frac{1}{2}$

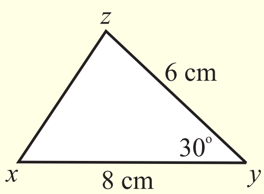
Sine is positive in the first and second quadrants. Therefore, there are two possible values for the angle.



$\angle xyz = 30^\circ$ [First quadrant]

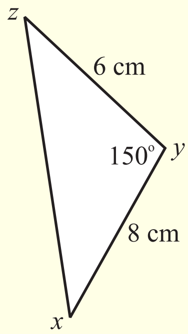
$= 150^\circ$ [Second quadrant]

5 (b) (ii) $a^2=b^2+c^2-2bc\cos A$ **19**



First triangle: $|xz|^2=8^2+6^2-2(8)(6)\cos 30^\circ$

$\Rightarrow |xz| = 4\cdot 1$ cm



Second triangle: $|xz|^2=8^2+6^2-2(8)(6)\cos 150^\circ$

$\Rightarrow |xz| = 13\cdot 5$ cm