TRIGONOMETRY (Q 4 & 5, PAPER 2)

2011

4. (a) Evaluate $\lim_{x\to\infty}\left(\frac{\sin 2x+\sin x}{3x}\right)$.

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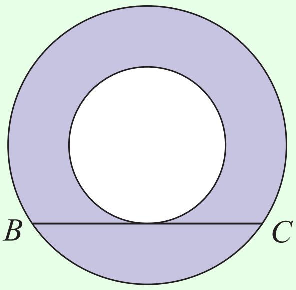
(b) Find all the solutions of the equation

$$\sin 2x+\cos x=0,\ \text{where}\ 0^\circ\le x\le 360^\circ.$$

(c) The diagram shows two concentric circles.
A tangent to the inner circle cuts the outer circle at B and C, where $|BC| = 2x$.



(i) Express the area of the shaded region in terms of x.

(ii) In the case where the radius of the outer circle is $2x$, show that the portion of the shaded region that lies below BC has area $\left(\frac{2\pi}{3}-\sqrt{3}\right)x^2$.

SOLUTION

4 (a)

$$\lim_{x\to 0}\left(\frac{\sin 2x+\sin x}{3x}\right)$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{3x}\right)+\lim_{x\to 0}\left(\frac{\sin x}{3x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\times\frac{2x}{3x}\right)+\lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{x}{3x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)\times\frac{2}{3}+\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\frac{1}{3}$$

$$=1\times\frac{2}{3}+1\times\frac{1}{3}$$

$$=1$$

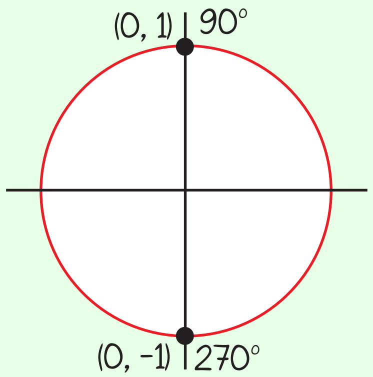
4 (b)

$\sin 2x + \cos x = 0$

$2\sin x \cos x + \cos x = 0$ $\quad$ $\boxed{\sin 2A = 2\sin A \cos A}$

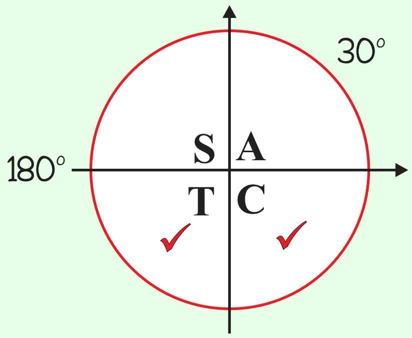
$\cos x(2\sin x + 1) = 0$

$\cos x = 0$



$x = 90^\circ, 270^\circ$

$2\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2}$



$x = 210^\circ, 330^\circ$

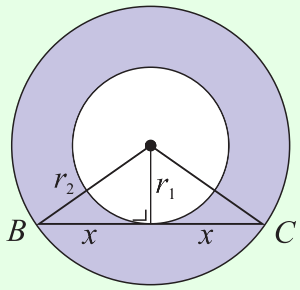
Answer: $x = 90^\circ, 210^\circ, 270^\circ, 330^\circ$

4 (c)

Call r_1, the radius of the smaller circle and r_2, the radius of the bigger circle.

$\therefore r_2^2 = r_1^2 + x^2$

$r_2 = \sqrt{r_1^2 + x^2}$



4 (c) (i)

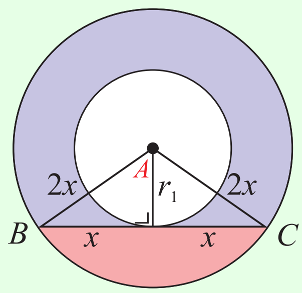
Area of shaded region = Area of bigger circle – Area of smaller circle

$$= \pi r_2^2 - \pi r_1^2$$
$$= \pi(\sqrt{r_1^2 + x^2})^2 - \pi r_1^2$$
$$= \pi(r_1^2 + x^2) - \pi r_1^2$$
$$= \pi r_1^2 + \pi x^2 - \pi r_1^2$$
$$= \pi x^2$$

4 (c) (ii)

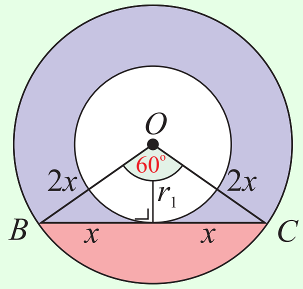
Calculate the angle A shown.

$$\sin A = \frac{x}{2x} = \frac{1}{2} \Rightarrow A = \sin^{-1}(\tfrac{1}{2}) = 30^\circ$$



Call the centre of the circles O.

$\therefore |\angle BOC| = 60^\circ$



Area of portion below BC

= Area of sector OBC – Area of triangle OBC

$A = \frac{1}{2}r^2\theta$ $\quad$ $A = \frac{1}{2}ab\sin C$

$= \frac{1}{2}(2x)^2(\frac{\pi}{3}) - \frac{1}{2}(2x)(2x)\sin 60^\circ$

$= \frac{1}{2}(4x^2)(\frac{\pi}{3}) - 2x^2(\frac{\sqrt{3}}{2})$

$= 2x^2(\frac{\pi}{3}) - \sqrt{3}x^2$

$= \left(\frac{2\pi}{3} - \sqrt{3}\right)x^2$

5. **(a)** Find the values of x for which $3\tan x = \sqrt{3}$, where $0^\circ \le x \le 360^\circ$.

(b) **(i)** Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

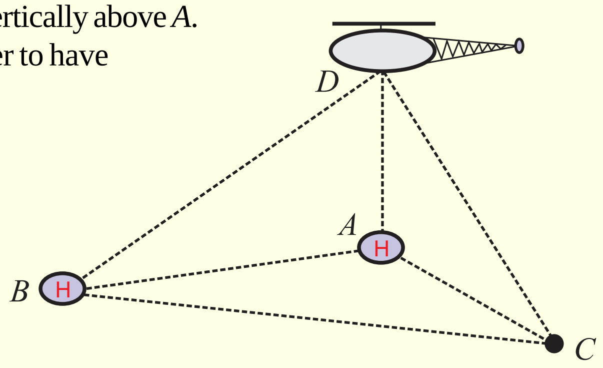
(ii) Show that if $\alpha + \beta = 90^\circ$, then $\dfrac{\tan 2\alpha}{\tan 2\beta} = -1$.

(c) A and B are two helicopter landing pads on level ground. C is another point on the same level ground. $|BC| = 800$ metres, $|AC| = 900$ metres, and $|\angle BCA| = 60^\circ$.

A helicopter at point D is hovering vertically above A.
A person at C observes the helicopter to have an angle of elevation of 30°.

(i) Find $|AD|$, in surd form.

(ii) Find $|BD|$.



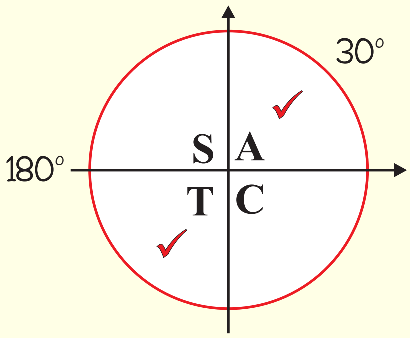
SOLUTION

5 (a)

$3\tan x = \sqrt{3}$

$\tan x = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \Rightarrow x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$

$x = 30^\circ, 210^\circ$



5 (b) (i)

LHS

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

RHS

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-(\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})}$$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-(\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})}\times\frac{\cos A\cos B}{\cos A\cos B}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$LHS=RHS$$

5 (b) (ii)

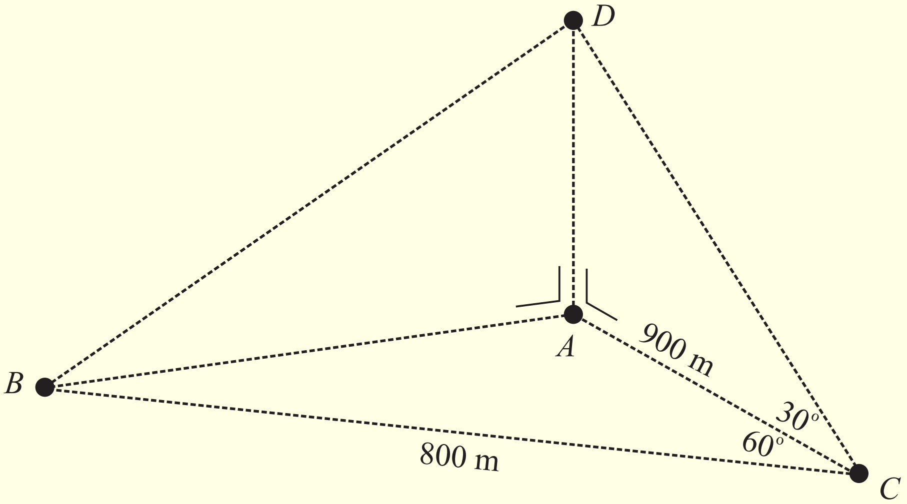
$$\alpha+\beta=90^\circ\Rightarrow\beta=(90^\circ-\alpha)$$

$$\frac{\tan 2\alpha}{\tan 2\beta}=\frac{\tan 2\alpha}{\tan 2(90^\circ-\alpha)}=\frac{\tan 2\alpha}{\tan(180^\circ-2\alpha)}$$

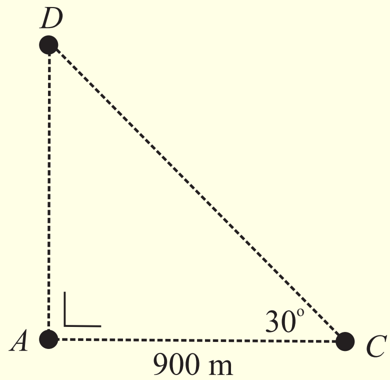
$$\tan(180^\circ-2\alpha)=\frac{\tan 180^\circ-\tan 2\alpha}{1+\tan 180^\circ\tan 2\alpha}=\frac{0-\tan 2\alpha}{1+(0)\tan 2\alpha}=-\tan 2\alpha$$

$$\therefore\frac{\tan 2\alpha}{\tan 2\beta}=\frac{\tan 2\alpha}{\tan(180^\circ-2\alpha)}=\frac{\tan 2\alpha}{-\tan 2\alpha}=-1$$

5 (c) (i)



Lift out the right-angled triangle *DAC*:



$$\tan 30^\circ=\frac{|AD|}{900}$$

$$\therefore|AD|=900\tan 30^\circ$$

$$=900\left(\frac{1}{\sqrt{3}}\right)=300\sqrt{3}\text{ m}$$

5 (c) (ii)

Apply the Cosine rule to triangle ABC:

$$|AB|^2 = 900^2 + 800^2 - 2(900)(800)\cos 60^\circ$$
$$= 810000 + 640000 - 2(900)(800)(\tfrac{1}{2})$$
$$= 810000 + 640000 - 720000$$
$$= 730000$$
$$\therefore |AB| = \sqrt{730000} \text{ m}$$

$a^2 = b^2 + c^2 - 2bc\cos A$

Lift out the right-angled triangle ABD and apply Pythagoras:

$$|BD|^2 = |AB|^2 + |AD|^2$$
$$= (\sqrt{730000}) + (300\sqrt{3})^2$$
$$= 730000 + 90000 \times 3$$
$$= 730000 + 270000$$
$$= 1000000$$
$$\therefore |BD| = \sqrt{1000000} = 1000 \text{ m}$$