TRIGONOMETRY (Q 4 & 5, PAPER 2)

2010

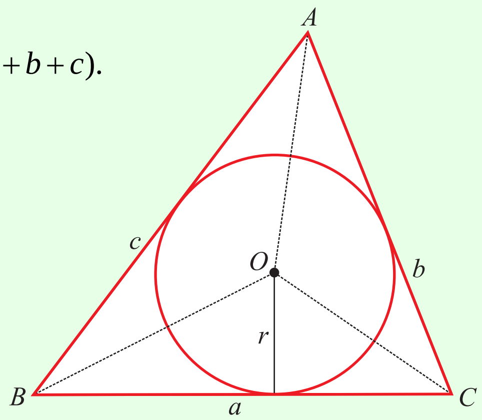
4 (a) The area of a triangle PQR is 20 cm². $|PQ| = 10$ cm and $|PR| = 8$ cm.

Find the two possible values of $|\angle QPR|$.

(b) Find all the solutions of the equation $\cos 2x = \cos x$ in the domain $0^\circ \leq x \leq 360^\circ$.

(c) ABC is a triangle with sides of lengths a, b and c, as shown.
Its incircle has centre O and radius r.

(i) Show that the area of ΔABC is $\frac{1}{2}r(a+b+c)$.



(ii) The lengths of the sides of a triangle are $a = p^2 + q^2$, $b = p^2 - q^2$, and $c = 2pq$, where p and q are natural numbers and $p > q$.
Show that this triangle is right-angled.

(iii) Show that the radius of the incircle of the triangle in part (ii) is a whole number.

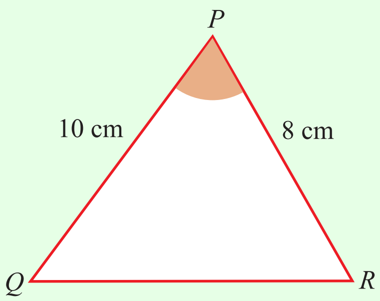
SOLUTION

4 (a)

$20 = \frac{1}{2}(10)(8)\sin|\angle QPR|$

$\sin|\angle QPR| = \frac{1}{2}$

$|\angle QPR| = \sin^{-1}(\frac{1}{2}) = 30^\circ, 150^\circ$



$A = \frac{1}{2}ab\sin C$

4 (b)

$\cos 2x = \cos x$

$\cos 2x - \cos x = 0$

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$

$\cos^2 x - \sin^2 x - \cos x = 0$

$\boxed{\cos^2 A + \sin^2 A = 1}$

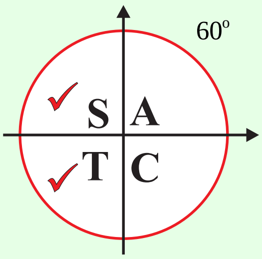
$\cos^2 x - (1 - \cos^2 x) - \cos x = 0$

$\cos^2 x - 1 + \cos^2 x - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

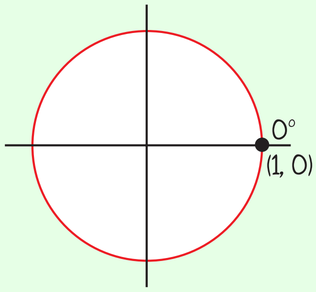
$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$



$x = 120°, 240°$

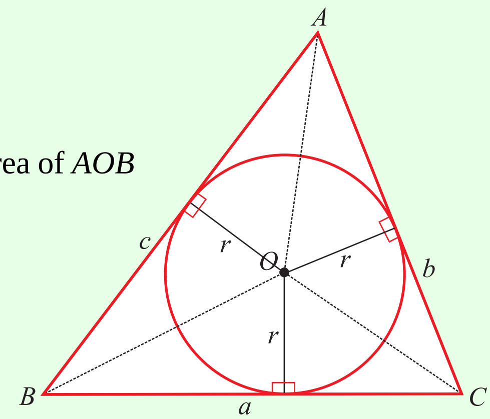
$\cos x = 1$



$x = 0°, 360°$

$x = 0°, 120°, 240°, 360°$

4 (c) (i)



Area of ABC = Area of BOC + Area of COA + Area of AOB

$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

$= \frac{1}{2}r(a+b+c)$

4 (c) (ii)

$a = p^2 + q^2$ [Longest side]

$\boxed{x^2 + y^2 = r^2}$

$b = p^2 - q^2$

$c = 2pq$

$a^2 = (p^2+q^2)^2 = p^4 + 2p^2q^2 + q^4$

$b^2 + c^2 = (p^2 - q^2)^2 + (2pq)^2$

$= p^4 - 2p^2q^2 + q^4 + 4p^2q^2$

$= p^4 + 2p^2q^2 + q^4$

$\therefore a^2 = b^2 + c^2$

4 (c) (ii)

Area of $\Delta ABC = \frac{1}{2}(2pq)(p^2 - q^2) = pq(p^2 - q^2)$ [Right-angled triangle: Area is equal to half the base by the height.]

Area of $\Delta ABC = \frac{1}{2}r(p^2 + q^2 + p^2 - q^2 + 2pq)$ [Using the result in part (i).]

$$= \frac{1}{2}r(2p^2 + 2pq)$$

$$= r(p^2 + pq)$$

$$\therefore pq(p^2 - q^2) = r(p^2 + pq)$$

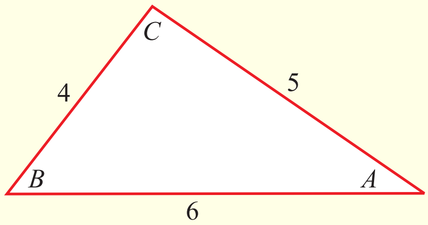
$$pq(p+q)(p-q) = rp(p+q)$$

$$q(p-q) = r$$

p and q are natural numbers. $p > q$ which means $(p - q)$ is a natural number. Therefore, r is a whole number.

5 (a) Given that $\tan\theta = \frac{1}{3}$, show that $\tan 2\theta = \frac{3}{4}$.

(b) A triangle has sides of lengths 4, 5 and 6.
The angles of the triangle are A, B and C, as in the diagram.



(i) Using the cosine rule, show that $\cos A + \cos C = \frac{7}{8}$.

(ii) Show that $\cos(A+C) = -\frac{9}{16}$.

(c) (i) Show that $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2 + 2\cos(A - B)$.

(ii) Hence solve the equation $(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2 + 2\sqrt{3}\sin 3x$ in the domain $0^\circ \le x \le 360^\circ$.

SOLUTION

5 (a)

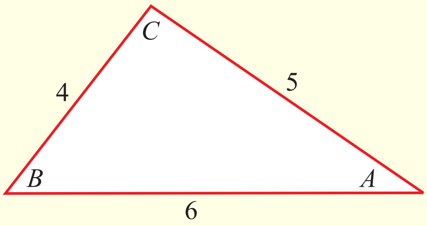
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2(\frac{1}{3})}{1 - (\frac{1}{3})^2}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$= \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

5 (b) (i)

$a^2 = b^2 + c^2 - 2bc\cos A$



$4^2 = 5^2 + 6^2 - 2(5)(6)\cos A$

$16 = 25 + 36 - 60\cos A$

$60\cos A = 45$

$\cos A = \frac{45}{60} = \frac{3}{4}$

$6^2 = 4^2 + 5^2 - 2(4)(5)\cos C$

$36 = 16 + 25 - 40\cos C$

$40\cos C = 5$

$\cos C = \frac{5}{40} = \frac{1}{8}$

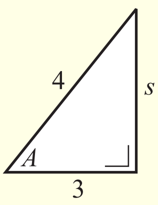
$\therefore \cos A + \cos C = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

5 (b) (ii)

$\cos(A + C) = \cos A\cos C - \sin A\sin C$

$= \frac{3}{4} \times \frac{1}{8} - \frac{\sqrt{7}}{4} \times \frac{\sqrt{63}}{8}$

$= \frac{3}{32} - \frac{21}{32} = -\frac{18}{32} = -\frac{9}{16}$

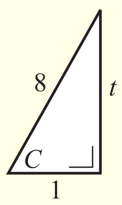


$4^2 = 3^2 + s^2$

$16 - 9 = s^2$

$7 = s^2$

$\sqrt{7} = s$



$8^2 = 1^2 + t^2$

$64 - 1 = t^2$

$63 = t^2$

$\sqrt{63} = t$

5 (c) (i)

$\cos^2 A + \sin^2 A = 1$

COMPOUND ANGLES
$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$(\cos A + \cos B)^2 + (\sin A + \sin B)^2$

$= \cos^2 A + 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B$

$= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B + 2(\cos A\cos B + \sin A\sin B)$

$= 2 + 2\cos(A - B)$

5 (c) (ii)

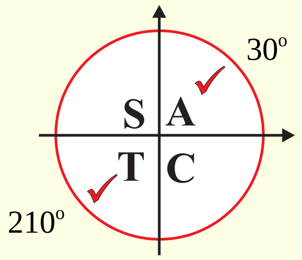
$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2 + 2\cos(4x - x) = 2 + 2\cos 3x$$

$$\therefore 2 + 2\cos 3x = 2 + 2\sqrt{3}\sin 3x$$

$$\cos 3x = \sqrt{3}\sin 3x$$

$$\frac{1}{\sqrt{3}} = \frac{\sin 3x}{\cos 3x}$$

$$\frac{1}{\sqrt{3}} = \tan 3x$$



$3x$ = 30°, 390°, 750° (First quadrant)
= 210°, 570°, 930° (Third quadrant)

x = 10°, 130°, 250° (First quadrant)
= 70°, 190°, 310° (Third quadrant)

,
x = 10°, 70°, 130°, 190°, 250°, 310°