Trigonometry (Q 4 & 5, Paper 2)

2008

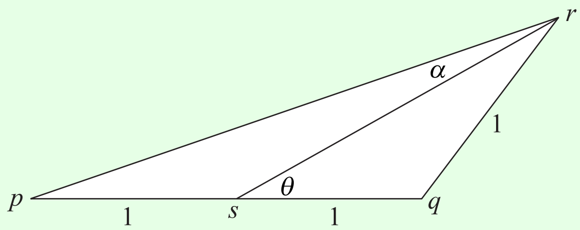
4 (a) A and B are acute angles such that $\tan A = \frac{5}{12}$ and $\tan B = \frac{3}{4}$.

Find $\cos(A-B)$ as a fraction.

(b) (i) Show that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$.

(ii) Hence, or otherwise, prove that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

(c) In the triangle pqr, $|\angle rsq| = \theta$, $|\angle prs| = \alpha$, $|rq| = 1$, $|ps| = 1$ and $|sq| = 1$.



(i) Find $|sr|$ in terms of θ.

(ii) Hence, or otherwise, show that $\tan\theta = 3\tan\alpha$.

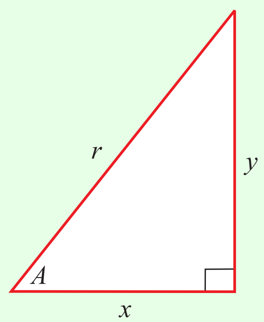
Solution

4 (a)

$\boxed{\cos A = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}}}$ **1**

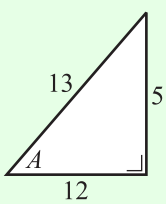
$\boxed{\sin A = \frac{y}{r} = \frac{\text{Opposite}}{\text{Hypotenuse}}}$ **2**

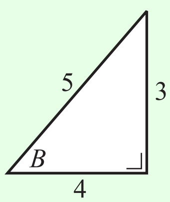
$\boxed{\tan A = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}}$ **3**



$\boxed{x^2 + y^2 = r^2}$ **4**

Draw two right-angled triangles. Use Pythagoras or your knowledge of famous triangles to put in the missing sides.





$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$ **12**

$\therefore \cos(A-B) = (\frac{12}{13})(\frac{4}{5}) + (\frac{5}{13})(\frac{3}{5}) = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$

4 (b) (i)

Steps
1. You have to prove that the left-hand side (*LHS*) equals the right-hand side (*RHS*).
2. Change everything to sine and cosine.
3. Simplify each side using page 9 of the tables and good algebra.
4. For half angles, $\frac{\theta}{2}$: Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$.

LHS

$$\frac{\sin 2A}{1+\cos 2A}$$
$$= \frac{2\sin A\cos A}{1+\cos^2 A - \sin^2 A}$$
$$= \frac{2\sin A\cos A}{2\cos^2 A}$$
$$= \frac{\sin A}{\cos A}$$

RHS

$$\tan A$$
$$= \frac{\sin A}{\cos A}$$

$\boxed{\cos^2 A + \sin^2 A = 1}$ **8**

$\boxed{\sin 2A = 2\sin A\cos A}$ **13**

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$ **14**

4 (b) (ii)

Let $A = 22\frac{1}{2}^{\circ} \Rightarrow 2A = 45^{\circ}$

$$\therefore \tan 22\tfrac{1}{2}^{\circ} = \frac{\sin 2(22\frac{1}{2}^{\circ})}{1+\cos 2(22\frac{1}{2}^{\circ})} = \frac{\sin 45^{\circ}}{1+\cos 45^{\circ}}$$

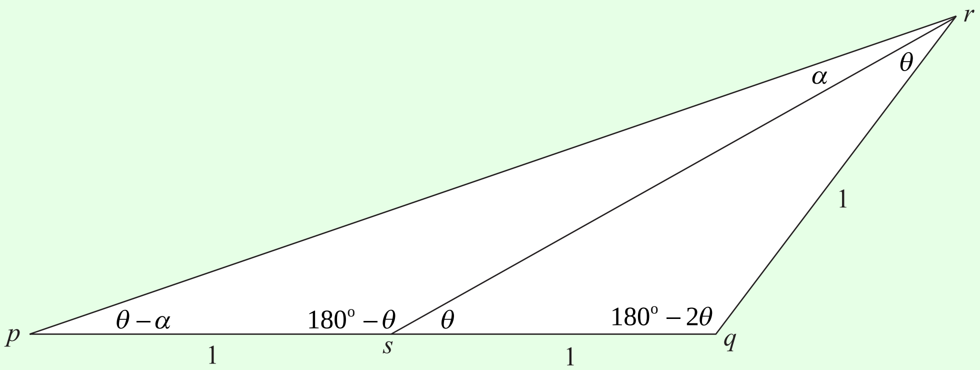
$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{1}{\sqrt{2}}}{(1+\frac{1}{\sqrt{2}})} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}+1}$$ [Multiply above and below by $\sqrt{2}$]

$$= \frac{1}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$$ [Multiply above and below by the conjugate of the bottom.]

$$= \frac{(\sqrt{2}-1)}{2-1} = \sqrt{2}-1$$

4 (c) (i)



Triangle *srq* is isosceles. Therefore, the angles opposite the equal sides are also equal. You can now find the third angle as all three angles add up to 180°.

$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$ **18**

$$\therefore \frac{|sr|}{\sin(180^\circ - 2\theta)} = \frac{1}{\sin\theta} \Rightarrow |sr| = \frac{\sin(180^\circ - 2\theta)}{\sin\theta}$$

$$\Rightarrow |sr| = \frac{\sin 180^\circ \cos 2\theta - \cos 180^\circ \sin 2\theta}{\sin\theta}$$

$$\Rightarrow |sr| = \frac{\sin 2\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta}$$

$$\therefore |sr| = 2\cos\theta$$

4 (c) (ii)

Consider triangle *prs*:

$$\frac{\sin(\theta - \alpha)}{|pr|} = \frac{\sin\alpha}{1} \Rightarrow \sin(\theta-\alpha) = |pr|\sin\alpha$$

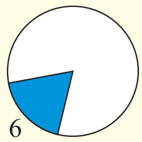
$$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 2\cos\theta\sin\alpha$$

$$\Rightarrow \sin\theta\cos\alpha = 3\cos\theta\sin\alpha$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = 3\frac{\sin\alpha}{\cos\alpha}$$

$$\therefore \tan\theta = 3\tan\alpha$$

5 (a) In the shaded sector in the diagram, the arc is 6 cm long, and the angle of the sector is 0.75 radians. Find the area of the sector.



(b) (i) Express $\sin 4x - \sin 2x$ as a product.

(ii) Find all the solutions of the equation

$$\sin 4x - \sin 2x = 0$$

in the domain $0^\circ \le x \le 180^\circ$.

(c) A triangle has sides of lengths *a*, *b* and *c*. The angle opposite the side of length *a* is *A*.

(i) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

(ii) If *a*, *b* and *c* are consecutive whole numbers, show that

$$\cos A = \frac{a+5}{2a+4}.$$

SOLUTION

5 (a)

$$r = \frac{s}{\theta} = \frac{6}{0.75} = 8 \text{ cm}$$

Arc length *s*: $\boxed{s = r\theta}$ **6**

$$\therefore A = \tfrac{1}{2}(8)^2(0.75) = 24 \text{ cm}^2$$

$\boxed{A = \tfrac{1}{2}r^2\theta}$ **7**

5 (b) (i)

$\sin 4x - \sin 2x$

$= 2\cos(\frac{4x+2x}{2})\sin(\frac{4x-2x}{2})$

$= 2\cos 3x \sin x$

SUMS → PRODUCTS

$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$

$\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

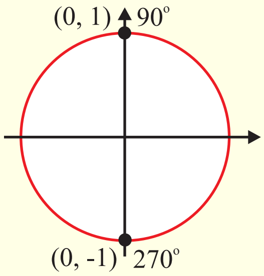
$\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$

5 (b) (ii)

$\sin 4x - \sin 2x = 0 \Rightarrow 2\cos 3x \sin x = 0$

$\therefore \cos 3x \sin x = 0$

$\therefore \cos 3x = 0$



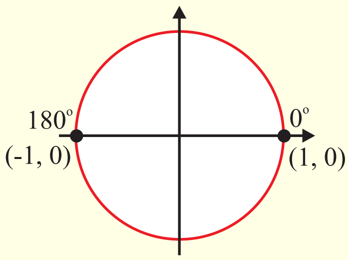
$3x \quad = 90^\circ, 450^\circ$

$\quad = 270^\circ, 630^\circ$

$x \quad = 30^\circ, 150^\circ$

$\quad = 90^\circ, \cancel{210^\circ}$

$\therefore \sin x = 0$



$x \quad = 0^\circ$

$\quad = 180^\circ$

Ans: 0°, 30°, 90°, 150°, 180°

5 (c) (i)

PROOF OF THE COSINE RULE

Applying Pythagoras to right-angled triangle **1**:

$a^2 = (c-x)^2 + h^2$

$\Rightarrow a^2 = (h^2 + x^2) + c^2 - 2cx$

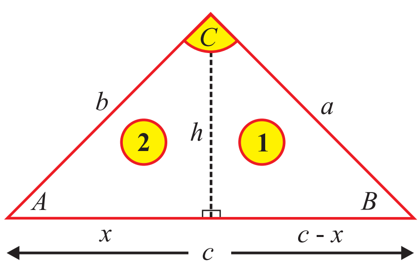
Applying Pythagoras to right-angled triangle **2**:

$b^2 = h^2 + x^2$

$\Rightarrow a^2 = b^2 + c^2 - 2cx$

Also $\cos A = \frac{x}{b} \Rightarrow x = b\cos A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$



5 (c) (ii)

a, b, c are consecutive numbers.
Call them a, $a + 1$, $a + 2$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore a^2 = (a+1)^2 + (a+2)^2 - 2(a+1)(a+2)\cos A$$

$$\Rightarrow a^2 = a^2 + 2a + 1 + a^2 + 4a + 4 - 2(a+1)(a+2)\cos A$$

$$\Rightarrow 2(a+1)(a+2)\cos A = a^2 + 6a + 5$$

$$\Rightarrow 2(a+1)(a+2)\cos A = (a+5)(a+1)$$

$$\Rightarrow \cos A = \frac{(a+5)\cancel{(a+1)}}{2\cancel{(a+1)}(a+2)}$$

$$\therefore \cos A = \frac{(a+5)}{2a+4}$$