TRIGONOMETRY (Q 4 & 5, PAPER 2)

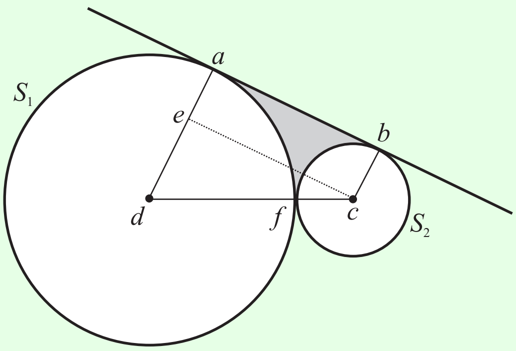
2005

4 (a) Evaluate $\lim_{\theta \to 0} \frac{\sin 4\theta}{3\theta}$.

4 (b) (i) Using $\cos 2A - \cos^2 A - \sin^2 A$, or otherwise, prove $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

(ii) Hence, or otherwise, solve the equation $1 + \cos 2x = \cos x$, where $0^\circ \le x \le 360^\circ$.

4 (c) S_1 is a circle of radius 9 cm and S_2 is a circle of radius 3 cm. S_1 and S_2 touch externally at *f*. A common tangent touches S_1 at point *a* and S_2 at *b*.



(i) Find the area of the quadrilateral *abcd*. Give your answer in surd form.

(ii) Find the area of the shaded region, which is bounded by [*ab*] and the monor arcs *af* and *bf*.

SOLUTION

4 (a)

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$ 20

$$\lim_{\theta \to 0} \frac{\sin 4\theta}{3\theta} = \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta} \times \frac{4\theta}{3\theta} = \frac{4}{3}$$

4 (b) (i)

STEPS
1. You have to prove that the left-hand side (*LHS*) equals the right-hand side (*RHS*).
2. Change everything to sine and cosine.
3. Simplify each side using page 9 of the tables and good algebra.
4. For half angles, $\frac{\theta}{2}$: Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$.

LHS

$\cos^2 A$

RHS

$\frac{1}{2}(1 + \cos 2A)$

$= \frac{1}{2}(1 + \cos^2 A - \sin^2 A)$

$= \frac{1}{2}(\cos^2 A + \cos^2 A)$ [Using $\cos^2 A + \sin^2 A = 1$]

$= \frac{1}{2}(2\cos^2 A) = \cos^2 A$

LHS = *RHS*

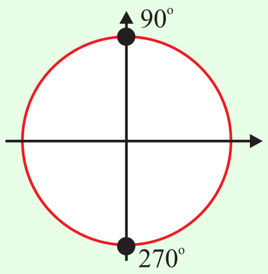
4 (b) (ii)

Solve $1+\cos 2x = \cos x$.

Use the previous result: $\cos^2 x = \frac{1}{2}(1+\cos 2x) \Rightarrow 2\cos^2 x = 1+\cos 2x$

$\therefore 2\cos^2 x = \cos x \Rightarrow 2\cos^2 x - \cos x = 0 \Rightarrow \cos x(2\cos x - 1) = 0$

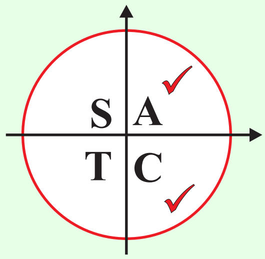
$\Rightarrow \cos x = 0$



$x = 90^\circ$

$= 270^\circ$

$\Rightarrow \cos x = \frac{1}{2}$

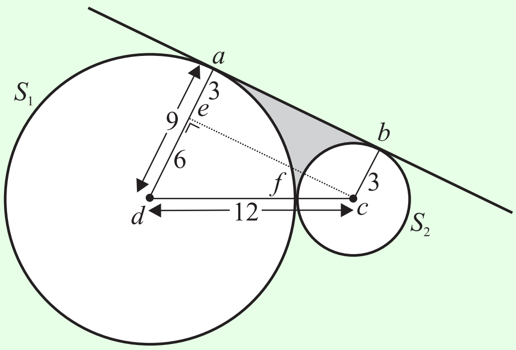


$x = 60^\circ$

$= 300^\circ$

Ans: $60^\circ, 90^\circ, 270^\circ, 300^\circ$

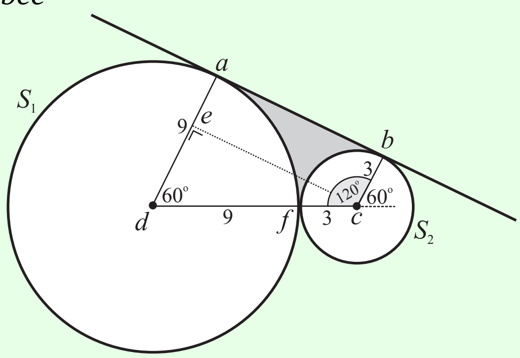
4 (c) (i)



Using Pythagoras on Δdec: $12^2 = 6^2 + |ec|^2 \Rightarrow |ec| = \sqrt{144-36} = \sqrt{108} = 6\sqrt{3}$

Area of $abcd$ = Area of Δdec + Area of rectangle $abce$

$= \frac{1}{2}(6)(6\sqrt{3}) + 3(6\sqrt{3}) = 18\sqrt{3} + 18\sqrt{3} = 36\sqrt{3}$



4 (c) (ii)

Before you do this, calculate some angles.

$\cos\angle edc = \frac{12}{6} = \frac{1}{2} \Rightarrow \angle edc = 60^\circ = \frac{\pi}{3}$

As $ad \parallel bc \Rightarrow \angle bcf = 180^\circ - 60^\circ = 120^\circ = \frac{2\pi}{3}$

Shaded area = Area of $abcd$ – Area of sector adf – Area of sector bcf

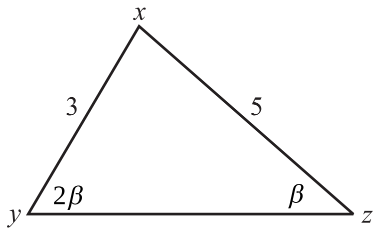
$A = \frac{1}{2}r^2\theta$ 7

Shaded area $= 36\sqrt{3} - \frac{1}{2}(9)^2(\frac{\pi}{3}) - \frac{1}{2}(3)^2(\frac{2\pi}{3}) = 36\sqrt{3} - \frac{27}{2}\pi - 3\pi$

$= 36\sqrt{3} - \frac{33}{2}\pi$

5 (a) The area of an equilateral triangle is $4\sqrt{3}$ cm^2. Find the length of a side of the triangle.

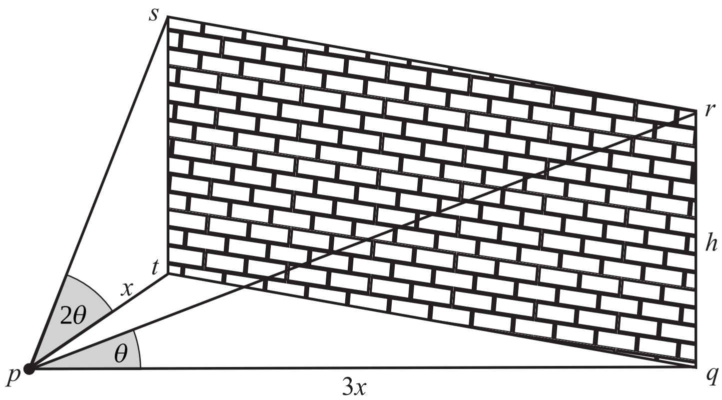
5 (b) In the triangle xyz, $|\angle xyz| = 2\beta$ and $|\angle xzy| = \beta$. $|xy| = 3$ and $|xz| = 5$.



(i) Use this information to express $\sin 2\beta$ in the form $\frac{a}{b}\sin\beta$, where $a, b \in \mathbf{N}$.

(ii) Hence express $\tan\beta$ in the form $\frac{\sqrt{c}}{d}$, where $c, d \in \mathbf{N}$.

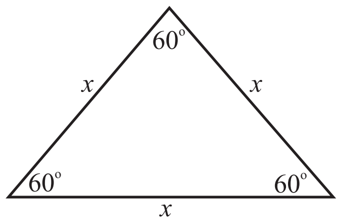
5 (c) $qrst$ is a vertical rectangular wall of height h on level ground. p is a point on the ground in front of the wall. The angle of elevation of r from p is θ and the angle of elevation of s from p is 2θ. $|pq| = 3|pt|$. Find θ.



SOLUTION

5 (a)

In an equilateral triangle, all sides and all angles are equal.



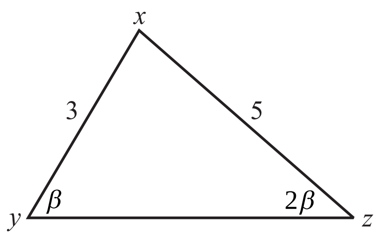
$A = \frac{1}{2}ab\sin C$ **5**

$A = \frac{1}{2}(x)(x)\sin 60^\circ = 4\sqrt{3}$

$\Rightarrow \frac{1}{2}x^2(\frac{\sqrt{3}}{2}) = 4\sqrt{3} \Rightarrow x^2 = 16 \Rightarrow x = 4$ cm

5 (b) (i)

Use the Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ **18**



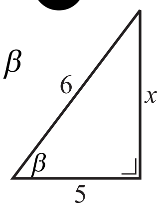
$\therefore \frac{\sin 2\beta}{5} = \frac{\sin\beta}{3} \Rightarrow \sin 2\beta = \frac{5}{3}\sin\beta$

5 (b) (ii)

$\sin 2A = 2\sin A\cos A$ **13**

$\sin 2\beta = \frac{5}{3}\sin\beta \Rightarrow 2\sin\beta\cos\beta = \frac{5}{3}\sin\beta$

$\Rightarrow 2\cos\beta = \frac{5}{3} \Rightarrow \cos\beta = \frac{5}{6}$



$\cos\theta = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ **1**

$x^2 + 5^2 = 6^2 \Rightarrow x^2 = 36 - 25 = 11$

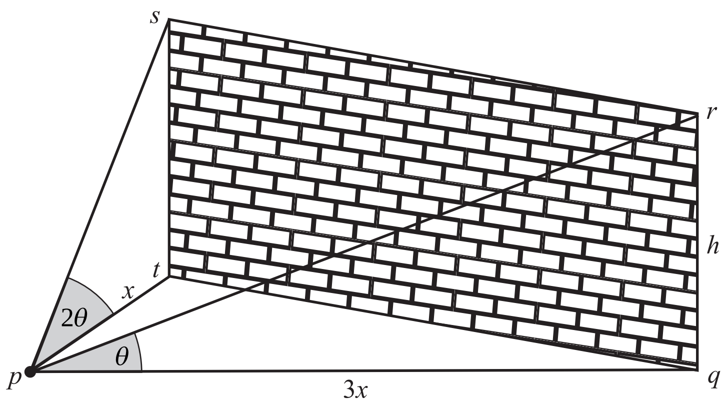
$\Rightarrow x = \sqrt{11}$

$\therefore \tan\beta = \frac{\sqrt{11}}{5}$

$\tan\theta = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ **3**

$x^2 + y^2 = r^2$ **4**

5 (c)



Consider Δprq: $\tan\theta = \frac{h}{3x} \Rightarrow h = 3x\tan\theta$....**(1)**

Consider Δspt: $\tan 2\theta = \frac{h}{x} \Rightarrow h = x\tan 2\theta$....**(2)**

Equate **(1)** and **(2)**: $\therefore 3x\tan\theta = x\tan 2\theta$

$\Rightarrow 3\tan\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Rightarrow 3 - 3\tan^2\theta = 2$

$\Rightarrow 1 = 3\tan^2\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$\tan\theta = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ **3**

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ **17**