TRIGONOMETRY (Q 4 & 5, PAPER 2)

2004

4 (a) A is an acute angle such that $\tan A = \frac{15}{17}$.

Without evaluating A, find

(i) $\cos A$

(ii) $\sin 2A$.

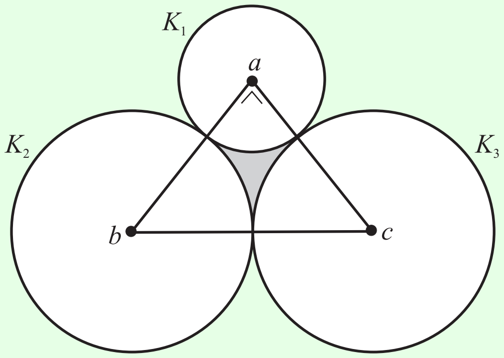
4 (b) (i) Prove that $\cos 2A = \cos^2 A - \sin^2 A$. Deduce that $\cos 2A = 2\cos^2 A - 1$.

(ii) Hence, or otherwise, find the value of θ for which $2\cos\theta - 7\cos(\frac{\theta}{2}) = 0$, where $0^\circ \leq \theta \leq 360^\circ$. Give your answer correct to the nearest degree.

4 (c) a, b and c are the centres of the circles K_1, K_2 and K_3 repectively. The three circles touch externally and $ab \perp ac$. K_2 and K_3 each have radius $2\sqrt{2}$ cm.

(i) Find, in surd form, the length of the radius of K_1.

(ii) Find the area of the shaded region in terms of π.



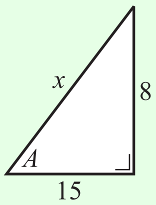
SOLUTION

4 (a)

Using Pythagoras:

$x^2 = 8^2 + 15^2 \Rightarrow x^2 = 64 + 225$

$\Rightarrow x^2 = 289 \Rightarrow x = 17$



$\cos\theta = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ 1

$\tan\theta = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$ 3

$x^2 + y^2 = r^2$ 4

$\sin 2A = 2\sin A\cos A$ 13

4 (a) (i)

$\cos A = \frac{15}{17}$

4 (a) (ii)

$\sin 2A = 2\sin A\cos A = 2 \times \frac{8}{17} \times \frac{15}{17} = \frac{240}{289}$

4 (b) (i)

Use formula **11** from page 9 of the tables and replace B by A.

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\Rightarrow \cos(A+A) = \cos A\cos A - \sin A\sin A$

$\Rightarrow \cos 2A = \cos^2 A - \sin^2 A$

Using formula 8 from page 9 of the tables:

$\cos^2 A + \sin^2 A = 1 \Rightarrow \sin^2 A = 1 - \cos^2 A$

$\therefore \cos 2A = \cos^2 A - (1 - \cos^2 A)$

$\Rightarrow \cos 2A = \cos^2 A - 1 + \cos^2 A \Rightarrow \cos 2A = 2\cos^2 A - 1$

COMPOUND ANGLES	
$\sin(A+B) = \sin A\cos B + \cos A\sin B$	 9
$\sin(A-B) = \sin A\cos B - \cos A\sin B$	 10
$\cos(A+B) = \cos A\cos B - \sin A\sin B$	 11
$\cos(A-B) = \cos A\cos B + \sin A\sin B$	 12

$\cos^2 A + \sin^2 A = 1$ 8

4 (b) (ii)

Half angles are the hardest with which to deal.
For a half angle: Let $\frac{x}{2}=A \Rightarrow x=2A$

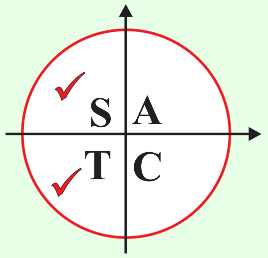
Let $\frac{\theta}{2}=A \Rightarrow \theta=2A$

$2\cos\theta-7\cos(\frac{\theta}{2})=0 \Rightarrow 2\cos 2A-7\cos A=0$

Use the previous result: $\therefore 2(2\cos^2 A-1)-7\cos A=0 \Rightarrow 4\cos^2 A-7\cos A-2=0$

$\Rightarrow (4\cos A+1)(\cos A-2)=0$

$\Rightarrow \cos A=-\frac{1}{4}$



$\cos^{-1}(0\cdot 25)=75\cdot 52^\circ$

$A=104\cdot 48^\circ$ [Second quadrant]

$=255\cdot 52^\circ$ [Third quadrant]

$\theta=2A=209^\circ$

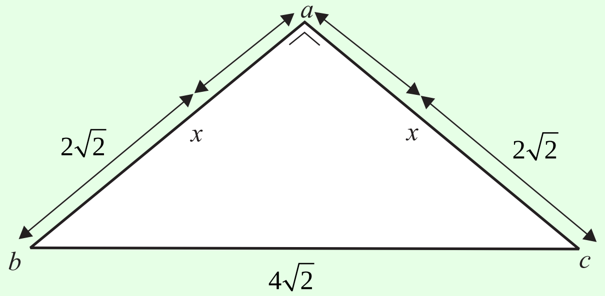
$=511^\circ$ [Outside the required range]

$\Rightarrow \cos A=2$

If you look up the inverse cos of 2 on your calculator you will get an error reading. There are no solutions to this equation.

Ans: 209°

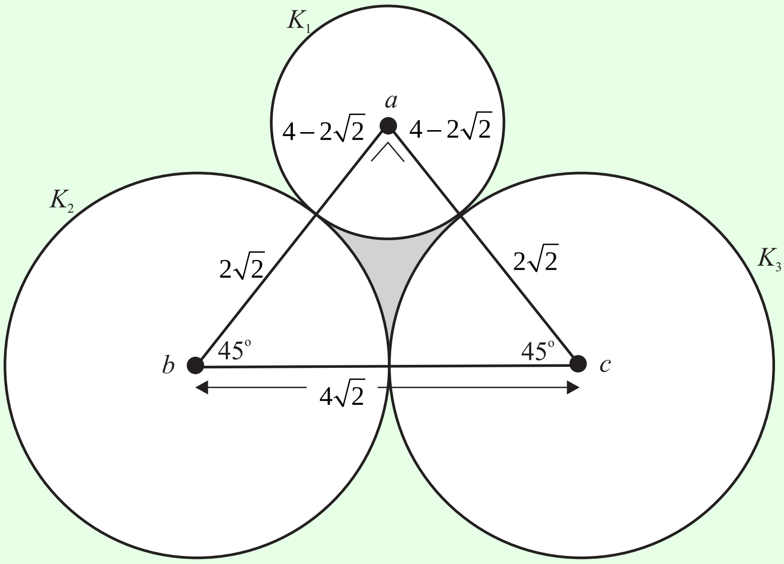
4 (c) (i)



Using Pythagoras on Δabc: $(4\sqrt{2})^2=x^2+x^2 \Rightarrow 32=2x^2 \Rightarrow x^2=16 \Rightarrow x=4$

Therefore, radius of K_1 is $4-2\sqrt{2}$.

4 (c) (ii)



Area of shaded region = Area of Δabc – (Area of the three sectors)

Area of a triangle: $A = \frac{1}{2} \times \text{Base} \times \text{Height}$

$A = \frac{1}{2} r^2 \theta$ **7**

$\therefore A = \frac{1}{2}(4)(4) - \frac{1}{2}(2\sqrt{2})^2(\frac{\pi}{4}) - \frac{1}{2}(2\sqrt{2})^2(\frac{\pi}{4}) - \frac{1}{2}(4-2\sqrt{2})^2(\frac{\pi}{2})$

$\therefore A = \frac{1}{2}(4)(4) - \frac{1}{2}(2\sqrt{2})^2(\frac{\pi}{4}) - \frac{1}{2}(2\sqrt{2})^2(\frac{\pi}{4}) - \frac{1}{2}(4-2\sqrt{2})^2(\frac{\pi}{2})$

$\Rightarrow A = 8 - \pi - \pi - (\frac{\pi}{4})(16 - 16\sqrt{2} + 8)$

$\Rightarrow A = 8 - 2\pi - 6\pi + 4\sqrt{2}\pi$

$\Rightarrow A = 8 - 8\pi + 4\sqrt{2}\pi$

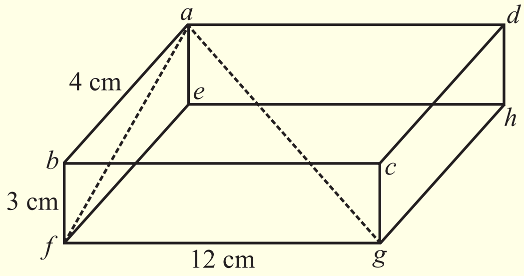
5 (a) Prove that $\cos^2 A + \sin^2 A = 1$, where $0^\circ \le A \le 90^\circ$.

5 (b) (i) Show that $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$ simplifies to a constant.

(ii) Express $1 - (\cos x - \sin x)^2$ in the form $a \sin bx$, where $a, b \in \mathbf{Z}$.

5 (c) The diagram shows a rectangular box. Rectangle *abcd* is the top of the box and rectangle *efgh* is the base of the box.

$|ab| = 4$ cm, $|bf| = 3$ cm and $|fg| = 12$ cm.



(i) Find $|af|$.

(ii) Find $|ag|$.

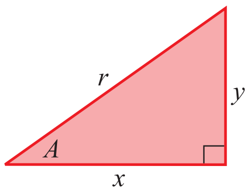
(iii) Find the measure of the acute angle between [*ag*] and [*df*]. Give your answer correct to the nearest degree.

SOLUTION

5 (a)

PROOF: $\cos A = \dfrac{x}{r} \Rightarrow \cos^2 A = \dfrac{x^2}{r^2}$

$\sin A = \dfrac{y}{r} \Rightarrow \sin^2 A = \dfrac{y^2}{r^2}$



$\therefore \cos^2 A + \sin^2 A = \dfrac{x^2 + y^2}{r^2} = \dfrac{r^2}{r^2} = 1 \Rightarrow \cos^2 A + \sin^2 A = 1$

5 (b) (i)

$(\cos x + \sin x)^2 + (\cos x - \sin x)^2 = \cos^2 x + 2\cos x \sin x + \sin^2 x + \cos^2 x - 2\cos x \sin x + \sin^2 x$

$= \cos^2 x + \sin^2 x + \cos^2 x + \sin^2 x = 1 + 1 = 2$

$\cos^2 A + \sin^2 A = 1$ **8**

5 (b) (ii)

$1 - (\cos x - \sin x)^2 = 1 - (\cos^2 x - 2\cos x \sin x + \sin^2 x)$

$= 1 - (1 - 2\cos x \sin x) = 2\cos x \sin x = \sin 2x$

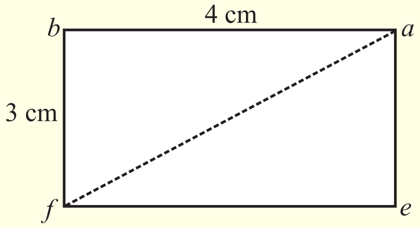
$\sin 2A = 2\sin A \cos A$ **13**

5 (c)

STEPS
1. Identify all right-angled triangles and non right-angled triangles and mark all angles and sides and label all vertices.
2. Separate out the triangles.

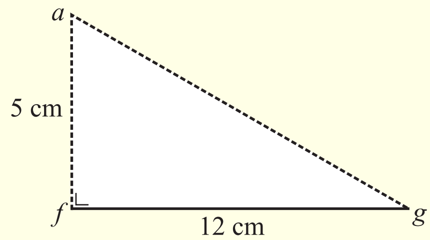
5 (c) (i)
Pick out rectangle *bfea*.

$|af|^2 = 3^2 + 4^2 = 25 \Rightarrow |af| = 5$ cm



5 (c) (ii)
Pick out Δafg.

$|ag|^2 = 12^2 + 5^2 = 169 \Rightarrow |ag| = 13$ cm



5 (c) (iii)
The diagonals bisect each other. Now, use the cosine rule to find θ.

$$a^2 = b^2 + c^2 - 2bc\cos A$$ **19**

$5^2 = 6\cdot5^2 + 6\cdot5^2 - 2(6\cdot5)(6\cdot5)\cos\theta$

$-59\cdot5 = -84\cdot5\cos\theta \Rightarrow \cos\theta = \dfrac{59\cdot5}{84\cdot5}$

$\Rightarrow \theta = 45^\circ$