TRIGONOMETRY (Q 4 & 5, PAPER 2)

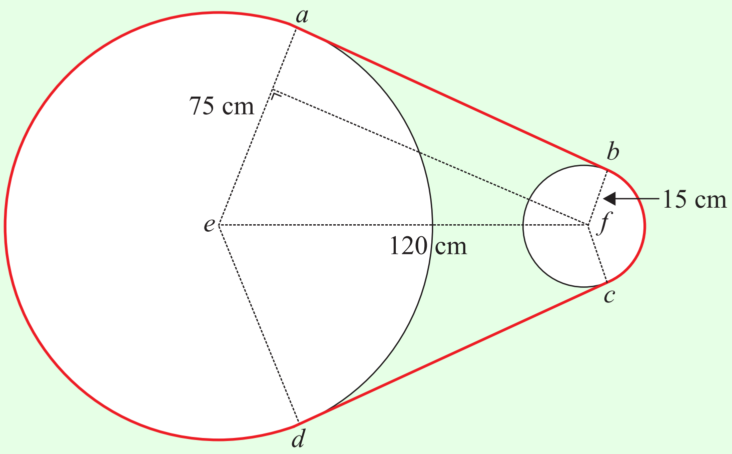
2002

4 (a) Find the value of θ for which $\cos\theta = -\frac{\sqrt{3}}{2}$, $0^\circ \le \theta \le 180^\circ$.

4 (b) (i) Use the formula $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ to express $\sin^2 \frac{1}{2}x$ in terms of cos x.

(ii) Hence, or otherwise, find all the solutions of the equation $\sin^2 \frac{1}{2}x - \cos^2 x = 0$ in the domain $0^\circ \le \theta \le 360^\circ$.

4 (c) A chain passes around two circular wheels as shown. One wheel has a radius 75 cm and the other has radius 15 cm. The centres, e and f, of the wheels are 120 cm apart. The chain consists of the common tangent [ab], the minor arc bc, the common tangent [cd] and the major arc da.



(i) Find the measure of $\angle aef$.

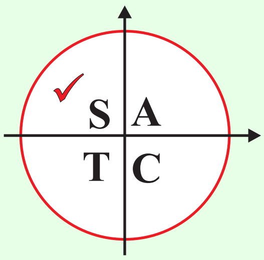
(ii) Find $|ab|$ in surd form.

(iii) Find the length of the chain, giving your answer in the form $k\pi + l\sqrt{3}$ where $k, l \in \mathbf{Z}$.

SOLUTION

4 (a)

The angle θ is in the second quadrant as cos is negative in this quadrant and it is within the range specified.
Find the basic angle from page 9 of the tables.



$\cos^{-1}(\frac{\sqrt{3}}{2}) = 30^\circ$

$\therefore \theta = 180^\circ - 30^\circ = 150^\circ$

4 (b) (i)

$\sin^2 A = \frac{1}{2}(1-\cos 2A)$

Let $A = \frac{1}{2}x \Rightarrow 2A = x$

$\therefore \sin^2 \frac{1}{2}x = \frac{1}{2}(1-\cos x)$

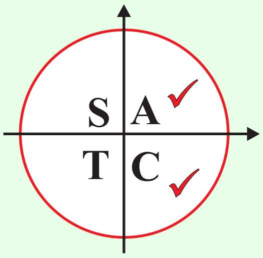
4 (b) (ii)

$\sin^2 \frac{1}{2}x - \cos^2 x = 0 \Rightarrow \frac{1}{2}(1-\cos x) - \cos^2 x = 0$

$\Rightarrow 1-\cos x - 2\cos^2 x = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

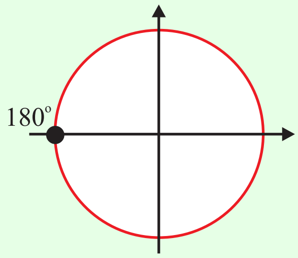
$\Rightarrow \cos x = \frac{1}{2}$



$x = 60^\circ$

$= 300^\circ$

$\Rightarrow \cos x = -1$



$x = 180^\circ$

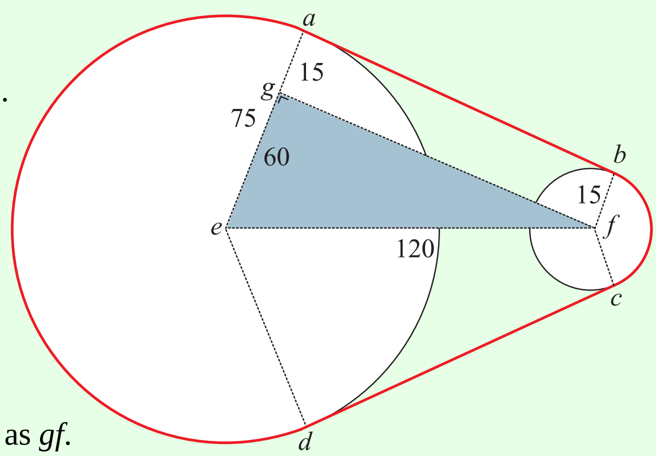
Ans: 60°, 180°, 300°

4 (c) (i)

Consider the blue right-angled triangle.

$\cos\theta = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ **1**

$\cos \angle aef = \frac{60}{120} = \frac{1}{2} \Rightarrow \angle aef = 60^\circ$



4 (c) (ii)

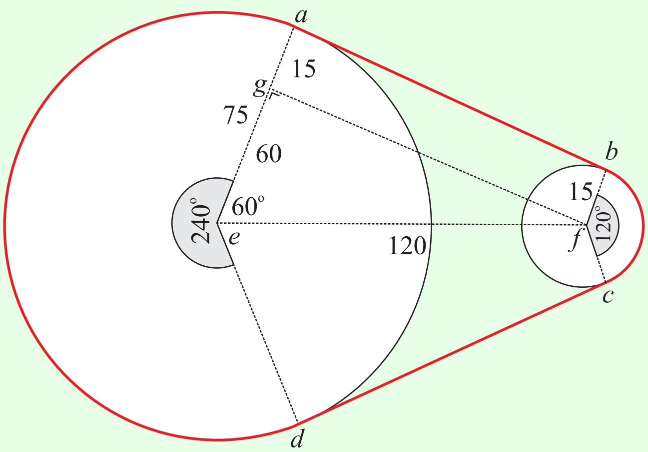
ab is parallel to and is the same length as *gf*.

$60^2 + |gf|^2 = 120^2 \Rightarrow |gf|^2 = 10,800$

$\Rightarrow |gf| = \sqrt{10,800} = \sqrt{3600 \times 3} = 60\sqrt{3} = |ab|$

4 (c) (iii)

Length of chain $L = |ab| + |cd| +$ Small arc bc + Large arc ad



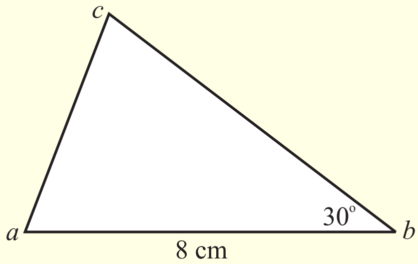
$\therefore L = 60\sqrt{3} + 60\sqrt{3} + 15(\frac{2\pi}{3}) + 75(\frac{4\pi}{3})$

$\Rightarrow L = 120\sqrt{3} + 10\pi + 100\pi = 120\sqrt{3} + 110\pi$

Arc length s

$s = r\theta$ **6**

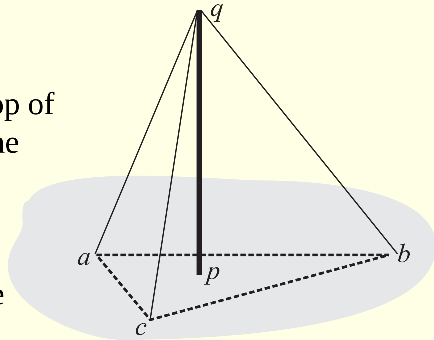
5 (a) The area of triangle abc is 12 cm^2. $|ab| = 8\text{ cm}$ and $|\angle abc| = 30^\circ$. Find $|bc|$.



5 (b) (i) Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

(ii) Hence, or otherwise, prove that $\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1$.

5 (c) A vertical radio mast $[pq]$ stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q, to the points a, b and c on the ground. The foot of the mast, p, lies inside the triangle abc.
Each cable is 52 m long and the mast is 48 m high.



(i) Find the (common) distance from p to each of the points a, b and c.

(ii) Given that $|ac| = 38$ m and $|ab| = 34$ m, find $|bc|$ correct to one decimal place.

Solution

5 (a)

$$A = \tfrac{1}{2}(8)(|bc|)\sin 30^\circ = 12 \Rightarrow 4(|bc|)\tfrac{1}{2} = 12$$

$$\Rightarrow |bc| = 6\text{ cm}$$

$A = \tfrac{1}{2}ab\sin C$ 5

5 (b) (i)

This is a trig identity.

Steps

1. You have to prove that the left-hand side (*LHS*) equals the right-hand side (*RHS*).
2. Change everything to sine and cosine.
3. Simplify each side using page 9 of the tables and good algebra.
4. For half angles, $\frac{\theta}{2}$: Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$.

LHS

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

RHS

$$\frac{\tan A + \tan B}{1 - \tan A\tan B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - (\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - (\frac{\sin A}{\cos A})(\frac{\sin B}{\cos B})} \times \frac{\cos A\cos B}{\cos A\cos B}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

LHS = *RHS*

5 (b) (ii)

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ **17**

Let $A = 22\tfrac{1}{2}^{\circ} \Rightarrow 2A = 45^{\circ}$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \Rightarrow \tan 45^{\circ} = \frac{2\tan A}{1-\tan^2 A}$$

$$\Rightarrow 1 = \frac{2\tan A}{1-\tan^2 A} \Rightarrow 1-\tan^2 A = 2\tan A$$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ **4**

$$\Rightarrow \tan^2 A + 2\tan A - 1 = 0$$

This is a quadratic equation that can be solved using formula **4**.

$a = 1,\ b = 2,\ c = -1$

$$\therefore \tan A = \frac{-2 \pm \sqrt{2^2-4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$$\Rightarrow \tan A = \frac{-2 \pm 2\sqrt{2}}{2} \Rightarrow \tan A = -1 \pm \sqrt{2}$$

$\tan A$ is in the first quadrant which means it is positive.

$$\therefore \tan A = \sqrt{2} - 1$$

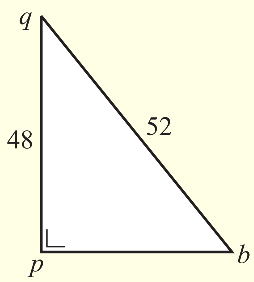
5 (c) (i)

Pick out the right-angled triangle Δqpb.

Using Pythagoras:

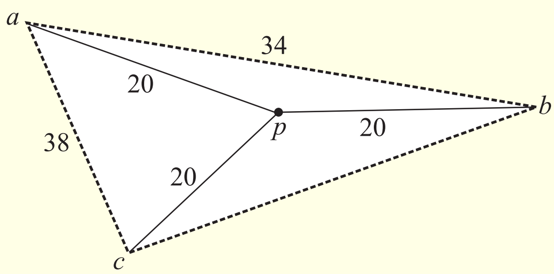
$$48^2 + |pb|^2 = 52^2 \Rightarrow |pb|^2 = 400 \Rightarrow |pb| = 20 \text{ m}$$

This is the common distance.



5 (c) (ii)

This requires the Cosine rule. You need to find angles in the first two triangles before you can solve the third triangle.



$a^2 = b^2 + c^2 - 2bc\cos A$ **19**

Consider Δapc: $38^2 = 20^2 + 20^2 - 2(20)(20)\cos\angle apc \Rightarrow \cos\angle apc = 143\cdot 6^{\circ}$

Consider Δapb: $34^2 = 20^2 + 20^2 - 2(20)(20)\cos\angle apb \Rightarrow \cos\angle apb = 116\cdot 4^{\circ}$

Consider Δcpb: $\angle cpb = 360^{\circ} - 143\cdot 6^{\circ} - 116\cdot 4^{\circ} = 100^{\circ}$

$$\therefore |bc|^2 = 20^2 + 20^2 - 2(20)(20)\cos 100^{\circ} \Rightarrow |bc| = 30\cdot 6 \text{ m}$$