Complex Numbers & Matrices (Q 3, Paper 1)

2007

3 (a) Let $A=\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 3 & \frac{3}{2}\end{pmatrix}$. Find A^2-2A.

(b) Let $z=-1+i$, where $i^2=-1$.

(i) Use De Moivre's theorem to evaluate z^5 and z^9.

(ii) Show that $z^5+z^9=12z$.

(c) (i) Find the two complex numbers $a+bi$ for which $(a+bi)^2=15+8i$.

(ii) Solve the equation $iz^2+(2-3i)z+(-5+5i)=0$.

Solution

3 (a)

$$A^2-2A=\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 3 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 3 & \frac{3}{2}\end{pmatrix}-2\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 3 & \frac{3}{2}\end{pmatrix}=\begin{pmatrix}1 & \frac{1}{2}\\ 6 & 3\end{pmatrix}-\begin{pmatrix}1 & \frac{1}{2}\\ 6 & 3\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

3 (b)

Polar form: $z=r(\cos\theta+i\sin\theta)$ 3

Changing from Cartesian to Polar

Steps

1. Find $r=|z|=\sqrt{\text{Re}^2+\text{Im}^2}$ first.

2. Draw a free-hand picture to see what quadrant θ is in.

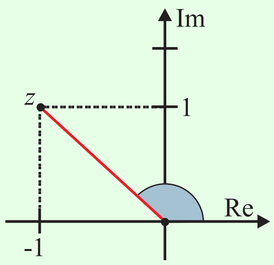
3. Find θ from $|\tan\theta|=\left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.

4. Write $z=r(\cos\theta+i\sin\theta)$. For **general** polar form add $2n\pi$ to θ.

Firstly, write $-1+i$ in polar form.

1. $r=\sqrt{1+1}=\sqrt{2}$

2. Diagram



3. $\tan\theta=\left|\frac{1}{-1}\right|=1\Rightarrow\theta=45^\circ$ in the first quadrant.

$\therefore\theta=135^\circ=\frac{3\pi}{4}$ in the second quadrant.

4. $z=\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

3 (b) (i)

$z^5 = 2^{\frac{5}{2}}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})^5 = 2^{\frac{5}{2}}(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4})$

$= 2^{\frac{5}{2}}(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = 4 - 4i$

$z^9 = 2^{\frac{9}{2}}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})^9 = 2^{\frac{9}{2}}(\cos\frac{27\pi}{4} + i\sin\frac{27\pi}{4})$

$= 2^{\frac{9}{2}}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = 16 - 16i$

STEPS
1. Write complex number in polar form.
2. Apply De Moivre's Theorem.
3. Change to Cartesian.

$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **4**

3 (b) (ii)

$z^5 + z^9 = 4 - 4i + 16 - 16i = 12 - 12i$

$= 12(1 - i) = 12z$

3 (c) (i)

1. $\sqrt{15 + 8i} = c + id$

2. $15 + 8i = (c^2 - d^2) + i2cd$

3. $(c^2 - d^2) = 15$ and $cd = 4$

4. $c = 4,\ d = 1$ or $c = -4,\ d = -1$

5. Ans: $\pm(4 + i)$

SQUARE ROOTS

STEPS

1. Put $\sqrt{a + ib} = c + id$.

2. Square: $a + ib = (c^2 - d^2) + i2cd$.

3. Equate the real and imaginary parts.

4. Solve simultaneously by guessing.

5. There are two answers $(\pm)$.

3 (c) (ii)

$a = i$
$b = (2 - 3i)$
$c = (-5 + 5i)$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **4**

$$z = \frac{-(2-3i) \pm \sqrt{(2-3i)^2 - 4i(-5+5i)}}{2i} = \frac{-2 + 3i \pm \sqrt{4 - 12i + 9i^2 + 20i - 20i^2}}{2i}$$

$$= \frac{-2 + 3i \pm \sqrt{15 + 8i}}{2i} = \frac{-2 + 3i \pm (4 + i)}{2i}$$ [Using result from **3 (c) (i)**]

$$= \frac{-2 + 3i + 4 + i}{2i}, \frac{-2 + 3i - 4 - i}{2i} = \frac{2 + 4i}{2i}, \frac{-6 + 2i}{2i} = \frac{1 + 2i}{i}, \frac{-3 + i}{i}$$

$= 2 - i,\ 1 + 3i$