COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

2006

3 (a) Given that $z = 2 + i,$ where $i^2 = -1,$ find the real number d such that $z + \frac{d}{z}$ is real.

3 (b) (i) Use matrix methods to solve the simultaneous equations

$$4x - 2y = 5$$
$$8x + 3y = -4$$

(ii) Find the two values of k which satisfy the matrix equation

$$\begin{pmatrix} 1 & k \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = 11$$

3 (c) (i) Express $-8 - 8\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta).$

(ii) Hence find $(-8 - 8\sqrt{3}i)^3.$

(iii) Find the four complex number z such that $z^4 = -8 - 8\sqrt{3}i.$ Give your answers in the form $a + bi,$ with a and b fully evaluated.

SOLUTION

3 (a)

$$z + \frac{d}{z} = 2 + i + \frac{d}{2+i}$$

$$\Rightarrow 2 + i + \frac{d}{(2+i)} \times \frac{(2-i)}{(2-i)} = 2 + i + \frac{2d - id}{5} = 2 + \frac{2d}{5} + i - \frac{id}{5}$$

As this number is real $\Rightarrow \left(1 - \frac{d}{5}\right)i = 0 \Rightarrow 1 = \frac{d}{5} \Rightarrow d = 5$

3 (b) (i)

Simultaneous equations in 2 or more unknowns can be written as a single matrix equation:

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$

The simultaneous equations can be written in matrix form as follows:

$$\begin{pmatrix} 4 & -2 \\ 8 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 3 & 2 \\ -8 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 7 \\ -56 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix}$$

ANSWER: $x = \frac{1}{4},\ y = -2$

3 (b)

$$(1 \quad k)\begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = 11 \Rightarrow (3-2k \quad k+4)\begin{pmatrix} 1 \\ k \end{pmatrix} = 11$$

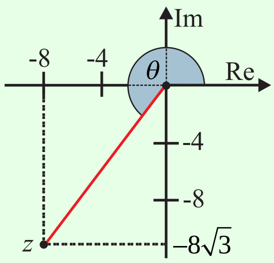
$$\Rightarrow 3-2k+k^2+4k = 11 \Rightarrow k^2+2k-8=0$$

$$\Rightarrow (k+4)(k-2)=0 \Rightarrow k=-4,\ 2$$

3 (c) (i)

STEPS

1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z = r(\cos\theta + i\sin\theta)$.



1. $r = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{64 + 3\times 64} = 16$
2. Draw picture.
3. $|\tan\theta| = \left|\dfrac{-8\sqrt{3}}{-8}\right| = \sqrt{3} \Rightarrow \theta = 60^\circ$

   Angle is in third quadrant $\Rightarrow \theta = 240^\circ = 240^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{4\pi}{3}$

4. $z = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

3 (c) (ii)

Use De Moivre's Theorem: $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **4**

$$z^3 = (-8-8\sqrt{3})^3 = [16(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3})]^3$$

$$= 16^3(\cos 4\pi + i\sin 4\pi) = 4096(1+0i) = 4096$$

3 (c) (iii)

To find roots, write z in general polar form.

$$z = 16\{\cos(\tfrac{4\pi}{3} + 2n\pi) + i\sin(\tfrac{4\pi}{3} + 2n\pi)\} = 16\left\{\cos\left(\frac{4\pi+6n\pi}{3}\right) + i\sin\left(\frac{4\pi+6n\pi}{3}\right)\right\}$$

$$z^{\frac{1}{4}} = 16^{\frac{1}{4}}\left\{\cos\left(\frac{4\pi+6n\pi}{3}\right) + i\sin\left(\frac{4\pi+6n\pi}{3}\right)\right\}^{\frac{1}{4}} = 2\left\{\cos\left(\frac{4\pi+6n\pi}{12}\right) + i\sin\left(\frac{4\pi+6n\pi}{12}\right)\right\}$$

$$\Rightarrow z^{\frac{1}{4}} = 2\left\{\cos\left(\frac{2\pi+3n\pi}{6}\right) + i\sin\left(\frac{2\pi+3n\pi}{6}\right)\right\}$$

- $n=0 \Rightarrow z_1 = 2\left\{\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right\} = 2\{\cos 60^\circ + i\sin 60^\circ\} = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 1+\sqrt{3}i$

- $n=1 \Rightarrow z_2 = 2\left\{\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right\} = 2\{\cos 150^\circ + i\sin 150^\circ\}$

  $= 2\{-\cos 30^\circ + i\sin 30^\circ\} = 2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = -\sqrt{3}+i$

- $n=2 \Rightarrow z_3 = 2\left\{\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right\} = 2\{\cos 240^\circ + i\sin 240^\circ\}$

  $= 2\{-\cos 60^\circ - i\sin 60^\circ\} = 2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = -1-\sqrt{3}i$

- $n=3 \Rightarrow z_3 = 2\left\{\cos\left(\frac{11\pi}{6}\right)+i\sin\left(\frac{11\pi}{6}\right)\right\} = 2\{\cos 330^\circ + i\sin 330^\circ\}$

  $= 2\{\cos 30^\circ - i\sin 30^\circ\} = 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = \sqrt{3}-i$

Answers: $1+\sqrt{3}i,\ -\sqrt{3}+i,\ -1-\sqrt{3}i,\ \sqrt{3}-i$