Complex Numbers & Matrices (Q 3, Paper 1)

2005

3 (a) Given that $A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, show that $A^3=A^{-1}$.

3 (b) Solve the quadratic equation $2iz^2+(6+2i)z+(3-6i)=0$, where $i^2=-1$.

3 (c) (i) $z=\cos\theta+i\sin\theta$. Use De Moivre's theorem to show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$, for $n\in\mathbf{N}$.

(ii) Expand $\left(z+\dfrac{1}{z}\right)^4$ and hence express $\cos^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$.

Solution

3 (a)

A is a diagonal matrix. Therefore, the following property holds:

$$D=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\Rightarrow D^n=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^n=\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix}$$ 7

The inverse of matrix *A* is given by:

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$ 8

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Rightarrow A^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^3=\begin{pmatrix}1^3 & 0\\ 0 & (-1)^3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

3 (b)

Solve $2iz^2+(6+2i)z+(3-6i)=0$

$a=2i$

$b=(6+2i)$

$c=(3-6i)$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(6+2i)\pm\sqrt{(6+2i)^2-4(2i)(3-6i)}}{4i}$$

$$=\frac{-(6+2i)\pm\sqrt{36+24i+4i^2-24i+48i^2}}{4i}=\frac{-(6+2i)\pm\sqrt{36-52}}{4i}$$

$$=\frac{-(6+2i)\pm\sqrt{-16}}{4i}=\frac{-6-2i\pm 4i}{4i}=\frac{-6+2i}{4i},\ \frac{-6-6i}{4i}$$

$$=\frac{-3+i}{2i},\ \frac{-3-3i}{2i}=\frac{1+3i}{2},\ \frac{-3+3i}{2}$$

3 (c) (i)

$$z^n + \frac{1}{z^n} = z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta)) = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

3 (c) (ii)

$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z^3\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + 4\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^4}\right)$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

Using the result in part (i):

$$\Rightarrow (2\cos\theta)^4 = (2\cos 4\theta) + 4(2\cos 2\theta) + 6$$

$$\Rightarrow 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\Rightarrow \cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$