Complex Numbers & Matrices (Q 3, Paper 1)

2004

3 (a) Find the real numbers p and q such that $2(p+iq)+i(p-iq)=5+i$, where $i^2=-1$.

3 (b) (i) $z_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ and $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$. Evaluate z_1z_2, giving your answer in the form $x+iy$.

(ii) $w_1=a+ib$ and $w_2=c+id$. Prove that $\overline{(w_1w_2)}=(\overline{w_1})(\overline{w_2})$, where $\overline{w}$ is the complex conjugate w.

3 (c) Let $A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$ and $P=\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}$.

(i) Evaluate $A^{-1}PA$ and hence $(A^{-1}PA)^{10}$.

(ii) Use the fact that $(A^{-1}PA)^{10}=A^{-1}P^{10}A$ to evaluate P^{10}.

Solution

3 (a)

$2(p+iq)+i(p-iq)=5+i \Rightarrow (2p-i^2q)+(p+2q)=5+i$

$\Rightarrow (2p+q)+(p+2q)=5+i$ [Equate the real parts and the imaginary parts.]

$\Rightarrow 2p+q=5$ and $p+2q=1$

Solving simultaneously: $p=3,\ q=-1$

3 (b) (i)

Rules for combining objects in Polar form

1. $(\cos A\oplus i\sin A)(\cos B\oplus i\sin B)=\cos(A+B)+i\sin(A+B)$

$z_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ and $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$.

$$\Rightarrow z_1z_2=\cos\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)=\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$$

$$=\cos 300^\circ+i\sin 300^\circ=\cos 60^\circ-i\sin 60^\circ=\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i$$

3 (b) (ii)

LHS

$\overline{(w_1w_2)}=\overline{(a+ib)(c+id)}$

$=\overline{(ac-bd)+(ad+bc)i}$

$=(ac-bd)-(ad+bc)i$

RHS

$(\overline{w_1})(\overline{w_2})=(\overline{a+ib})(\overline{c+id})$

$=(a-ib)(c-id)$

$=(ac-bd)-(ad+bc)i$

3 (c) (i)

$A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$ and $P=\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}$.

Evaluate $A^{-1}PA$ and hence $(A^{-1}PA)^{10}$.

$$D=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\Rightarrow D^n=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^n=\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix} \quad \ldots\ldots\ \mathbf{7}$$

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{(ad-bc)}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} \quad \ldots\ldots\ \mathbf{8}$$

$$A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{2-3}\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}=\frac{1}{-1}\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}$$

$$A^{-1}PA=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

The result is a diagonal matrix which means you can use formula 7.

$$(A^{-1}PA)^{10}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}^{10}=\begin{pmatrix}1^{10} & 0\\ 0 & 2^{10}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}$$

3 (c) (ii)

You are told that $(A^{-1}PA)^{10}=A^{-1}P^{10}A$ and are asked to find P^{10}.

$$\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}=A^{-1}P^{10}A\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}=\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}P^{10}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$$

To isolate P^{10}, multiply each side by matrix A at the front and A^{-1} at the end.

Therefore, $(A^{-1}PA)^{10}=A^{-1}P^{10}A\Rightarrow A(A^{-1}PA)^{10}A^{-1}=AA^{-1}P^{10}AA^{-1}$

$$\Rightarrow A(A^{-1}PA)^{10}A^{-1}=P^{10}$$

$$\Rightarrow P^{10}=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1024\end{pmatrix}\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -3072\\ -1 & 2048\end{pmatrix}\begin{pmatrix}-2 & -3\\ -1 & -1\end{pmatrix}=\begin{pmatrix}3070 & 3069\\ -2046 & -2045\end{pmatrix}$$