COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

2002

3 (a) Express $-1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.

3 (b) (i) Given that $z = 2 - i\sqrt{3}$, find the real number t such that $z^2 + tz$ is real.

(ii) w is a complex number such that $w\bar{w} - 2iw = 7 - 4i$, where $\bar{w}$ is the complex conjugate of w.
Find two possible values of w. Express each in the form $p + qi$, where $p, q \in \mathbf{R}$.

3 (c) The following three statements are true whenever x and y are real numbers:

- $x + y = y + x$
- $xy = yx$
- If $xy = 0$ then either $x = 0$ or $y = 0$.

Investigate whether the statements are also true when x is the matrix $\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$ and

y is the matrix $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$.

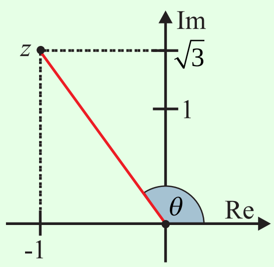
SOLUTION

3 (a)

STEPS
1. Find $r = |z| = \sqrt{\text{Re}^2 + \text{Im}^2}$ first.
2. Draw a free-hand picture to see what quadrant θ is in.
3. Find θ from $|\tan\theta| = \left|\frac{\text{Im}}{\text{Re}}\right|$ and by looking at the picture.
4. Write $z = r(\cos\theta + i\sin\theta)$.

1. $z = -1+\sqrt{3}i \Rightarrow r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

2. Draw a picture.



3. $\tan\theta = \left|\frac{\sqrt{3}}{-1}\right| = \sqrt{3} \Rightarrow \theta = 60^\circ$ (First quadrant)

$\therefore \theta = 120^\circ = \frac{2\pi}{3}$ (Second quadrant)

4. $\therefore z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

3 (b) (i)

$z = 2 - i\sqrt{3} \Rightarrow z^2 + tz = (2 - i\sqrt{3})^2 + t(2 - i\sqrt{3})$

$= 4 - 4\sqrt{3}i + 3i^2 + 2t - it\sqrt{3} = (1 + 2t) - (4\sqrt{3} + t\sqrt{3})i$

As this number is real $\Rightarrow 4\sqrt{3} + t\sqrt{3} = 0 \Rightarrow t = -4$

3 (b) (ii)

Let $w = p + qi$ and $\bar{w} = p - qi.$

$w\bar{w} - 2iw = 7 - 4i \Rightarrow (p + qi)(p - qi) - 2i(p + qi) = 7 - 4i$

$\Rightarrow p^2 + q^2 - 2ip - 2qi^2 = 7 - 4i \Rightarrow (p^2 + q^2 + 2q) - 2ip = 7 - 4i$

Equate the real and imaginary parts $\Rightarrow (p^2 + q^2 + 2q) = 7$ and $-2p = -4 \Rightarrow p = 2$

$\Rightarrow (2)^2 + q^2 + 2q = 7 \Rightarrow q^2 + 2q - 3 = 0 \Rightarrow (q + 3)(q - 1) = 0$

$\Rightarrow q = -3, 1$

Answers: $2 - 3i,\ 2 + i$

3 (c)

- $x + y = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix}$ and $y + x = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix}$

  $\therefore x + y = y + x$ (True)

- $xy = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $yx = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} -12 & 4 \\ -36 & 12 \end{pmatrix}$

  $\therefore xy \neq yx$ (False)

- If $xy = 0$ then either $x = 0$ or $y = 0$. This is false as it was already shown in the previous part $xy = 0$ even though $x \neq 0$ and $y \neq 0$.