COMPLEX NUMBERS & MATRICES (Q 3, PAPER 1)

2001

3 (a) Let $u = \dfrac{1+3i}{3+i}$ where $i^2 = -1$.

(i) Express u is the form $a + ib$ where $a, b \in \mathbf{R}$.

(ii) Evaluate $|u|$.

3 (b) (i) Write the simultaneous equations

$$x - \sqrt{3}y = -2$$
$$\sqrt{3}x + y = 2\sqrt{3}$$

in the form $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$ where A is a 2×2 matrix.

(ii) Then, find A^{-1} and use it to solve the equations for x and y.

3 (c) (i) Write $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $ax^2 + bxy + cy^2$ where $a, b, c \in \mathbf{Z}$.

(ii) Show that $z^2 - 16$ is a factor of $z^3 + (1+i)z^2 - 16z - 16(1+i)$ and hence, find the three roots of $z^3 + (1+i)z^2 - 16z - 16(1+i) = 0$.

SOLUTION

3 (a) (i)

$$u = \frac{1+3i}{3+i} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i} = \frac{3-i+9i-3i^2}{10} = \frac{6+8i}{10} = \frac{3+4i}{5}$$

$$= \tfrac{3}{5} + \tfrac{4}{5}i$$

3 (a) (ii)

$$u = \tfrac{3}{5} + \tfrac{4}{5}i \Rightarrow |u| = \sqrt{(\tfrac{3}{5})^2 + (\tfrac{4}{5})^2} = \sqrt{\tfrac{9}{25} + \tfrac{16}{25}} = \sqrt{\tfrac{25}{25}} = 1$$

3 (b) (i)

$$\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$$

3 (b) (ii)

$$A=\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}\Rightarrow A^{-1}=\frac{1}{1+3}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}-2\\ 2\sqrt{3}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}4\\ 4\sqrt{3}\end{pmatrix}=\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}$$

$\Rightarrow x=1$ and $y=\sqrt{3}$.

3 (c) (i)

$$(x \quad y)\begin{pmatrix}-2 & 3\\ -4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=(-2x-4y \quad 3x+5y)\begin{pmatrix}x\\ y\end{pmatrix}=(-2x^2-4xy+3xy+5y^2)$$

$$=(-2x^2-xy+5y^2)$$

3 (c) (ii)

To show that z^2-16 is a factor of $z^3+(1+i)z^2-16z-16(1+i)$, divide it in.

$$\begin{array}{r|l} & z+(1+i) \\ \hline z^2-16 & z^3+(1+i)z^2-16z-16(1+i) \\ & \mp z^3 \qquad\qquad \pm 16z \\ \hline & (1+i)z^2 \qquad -16(1+i) \\ & \mp(1+i)z^2 \qquad \pm 16(1+i) \\ \hline & \qquad 0 \end{array}$$

As the remainder is zero, z^2-16 is a factor.

$$z^3+(1+i)z^2-16z-16(1+i)=(z^2-16)(z+(1+i))=(z+4)(z-4)(z+(1+i))=0$$

$$\Rightarrow z=-4,\ 4,\ -(1+i)$$