ALGEBRA (Q 1 & 2, PAPER 1)

2006

1 (a) Find the real number a such that for all $x \neq 9$, $\dfrac{x-9}{\sqrt{x}-3} = \sqrt{x} + a$.

1 (b) $f(x) = 3x^3 + mx^2 - 17x + n$, where m and n are constants. Given that $x-3$ and $x+2$ are factors of $f(x)$, find the value of m and the value of n.

1 (c) $x^2 - t$ is a factor of $x^3 - px^2 - qx + r$.

(i) Show that $pq = r$.

(ii) Express the roots of $x^3 - px^2 - qx + r = 0$ in terms of p and q.

SOLUTIONS

1 (a)

Turn $x-9$ in a difference of 2 squares and factorise.

$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x})^2-(3)^2}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = (\sqrt{x}+3)$$

Therefore, $a = 3$.

OR

Cross-multiply: $\dfrac{x-9}{\sqrt{x}-3} = \sqrt{x} + a \Rightarrow x - 9 = (\sqrt{x}-3)(\sqrt{x}+a)$

$\Rightarrow x - 9 = x + (a-3)\sqrt{x} - 3a$

Lining up coefficents: $\Rightarrow -9 = -3a \Rightarrow a = 3$

1 (b)

$x-3$ is a factor $\Rightarrow f(3) = 3(3)^3 + m(3)^2 - 17(3) + n = 0 \Rightarrow 9m + n = -30$.....(**1**)

$x+2$ is a factor $\Rightarrow f(-2) = 3(-2)^3 + m(-2)^2 - 17(-2) + n = 0 \Rightarrow 4m + n = -10$.....(**2**)

Solving equation **1** and **2** simultaneously: $m = -4,\ n = 6$

1 (c) (i)

METHOD 1: DIVISION PROCESS

$$\begin{array}{r|l} & x - p \\ \hline x^2 - t & x^3 - px^2 - qx + r \\ & \mp x^3 \qquad\qquad \pm tx \\ \hline & \quad -px^2 + (t-q)x + r \\ & \quad \mp px^2 \qquad\qquad \pm pt \\ \hline & \qquad\qquad (t-q)x + (r - pt) \end{array}$$

The remainder has to be zero, i.e. $0x + 0$.

$\therefore t - q = 0 \Rightarrow t = q$ and $r - pt = 0 \Rightarrow r = pt$

$\therefore r = pq$

1 (c) (ii)

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - p) = 0$

$\Rightarrow x = \pm\sqrt{t},\ p \Rightarrow x = \pm\sqrt{q},\ p$

METHOD 2: LINING UP

1 (c) (i)

A cubic is a quadratic multiplied by a linear. The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear. Also, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear.

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - \frac{r}{t})$

$\Rightarrow (x^3 - px^2 - qx + r) = x^3 - \frac{r}{t}x^2 - tx + r$

Lining up the coefficients: $\Rightarrow p = \frac{r}{t}$ and $q = t$.

$\therefore p = \frac{r}{q} \Rightarrow r = pq$

1 (c) (ii)

$(x^3 - px^2 - qx + r) = (x^2 - t)(x - p) = 0$

$\Rightarrow x = \pm\sqrt{t},\ p \Rightarrow x = \pm\sqrt{q},\ p$

2 (a) Solve the simultaneous equations:

$$y = 2x - 5$$
$$x^2 + xy = 2$$

2 (b) (i) Find the range of values of $t \in \mathbf{R}$ for which the quadratic equation

$(2t-1)x^2 + 5tx + 2t = 0$ has real roots.

(ii) Explain why the roots are real when t is an integer.

2 (c) $f(x) = 1 - b^{2x}$ and $g(x) = b^{1+2x}$, where b is a positive real number. Find, in terms of b, the value of x for which $f(x) = g(x)$.

SOLUTION

2 (a)

$\frac{x}{5} - \frac{y}{4} = 0 \; (\times 20) \Rightarrow 4x - 5y = 0 \ldots. (\mathbf{1})$

$3x + \frac{y}{2} = 17 \; (\times 2) \Rightarrow 6x + y = 34 \ldots (\mathbf{2})$

$4x - 5y = 0 \ldots (\mathbf{1})$
$6x + y = 34 \ldots (\mathbf{2}) \; (\times -5)$

$4x - 5y \;\; = 0 \ldots (\mathbf{1})$
$30x + 5y = 170 \ldots (\mathbf{2})$
$34x \qquad = 170 \Rightarrow x = 5$

Substituting this value of x into equation **1** $\Rightarrow 4(5) - 5y = 0 \Rightarrow 20 = 5y \Rightarrow y = 4$

ANSWER: $x = 5$, $y = 4$

2 (b) (i)

$a = 2t - 1$
$b = 5t$
$c = 2t$

$(5t)^2 - 4(2t-1)(2t) \geq 0 \Rightarrow 9t^2 + 8t \geq 0$

Solve $9t^2 + 8t = 0 \Rightarrow t(9t + 8) = 0 \Rightarrow t = 0, -\frac{8}{9}$

Roots: $\alpha = -\frac{8}{9}$, $\beta = 0$

Region Test:

$\leftarrow$ $-\frac{8}{9}$	$-\frac{8}{9}$ $\leftrightarrow$ 0	0 $\rightarrow$
-1	$-\frac{1}{2}$	1
$9(-1)^2 + 8(-1) \geq 0$	$9(-\frac{1}{2})^2 + 8(-\frac{1}{2}) \geq 0$	$9(1)^2 + 8(1) \geq 0$
Correct	Wrong	Correct

Region Test on $9t^2 + 8t \geq 0$ .....**Test Box**

$\therefore t \leq -\frac{8}{9}, \; t \geq 0$

2 (b) (ii)

An integer is a whole number. The solutions are unreal for values of t in the range $-\frac{8}{9} \leq t \leq 0$. There are no integers in this range and therefore, the roots are real when t is an integer.

2 (c)

$f(x) = g(x) \Rightarrow 1 - b^{2x} = b^{1+2x}$

$\Rightarrow 1 = b^{1+2x} + b^{2x} \Rightarrow 1 = b^{2x}(b+1)$

$\Rightarrow \dfrac{1}{(b+1)} = b^{2x} \Rightarrow \log_{10}\left(\dfrac{1}{(b+1)}\right) = \log_{10} b^{2x}$

$\Rightarrow -\log_{10}(b+1) = 2x\log_{10} b$

$\Rightarrow -\dfrac{\log_{10}(b+1)}{2\log_{10} b} = x$

OR take the log to base *b*.

$\dfrac{1}{(b+1)} = b^{2x} \Rightarrow \log_b\left(\dfrac{1}{b+1}\right) = \log_b b^{2x}$

$\Rightarrow -\log_b(b+1) = 2x\log_b b$

$\Rightarrow -\frac{1}{2}\log_b(b+1) = x$

$\Rightarrow -\log_b \sqrt{b+1} = x$