ALGEBRA (Q 1 & 2, PAPER 1)

2005

1 (a) Solve the simultaneous equations:

$$\frac{x}{5}-\frac{y}{4}=0$$
$$3x+\frac{y}{2}=17$$

(b) (i) Express $2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}$ in the form $2^{\frac{p}{q}}$, where $p, q \in \mathbf{Z}$.

(ii) Let $f(x)=ax^3+bx^2+cx+d$. Show that $(x-t)$ is a factor of $f(x)-f(t)$.

(c) $(x-p)^2$ is a factor of x^3+qx+r. Show that $27r^2+4q^3=0$. Express the roots of $3x^2+q=0$ in terms of p.

SOLUTION

1 (a)

$\frac{x}{5}-\frac{y}{4}=0\ (\times 20) \Rightarrow 4x-5y=0....(\mathbf{1})$

$3x+\frac{y}{2}=17\,(\times 2) \Rightarrow 6x+y=34...(\mathbf{2})$

$4x-5y=0...(\mathbf{1})$
$6x+y=34...(\mathbf{2})\,(\times -5)$

$4x-5y\ =0...(\mathbf{1})$
$30x+5y=170...(\mathbf{2})$
$34x\quad =170 \Rightarrow x=5$

Substituting this value of x into equation **1** $\Rightarrow 4(5)-5y=0 \Rightarrow 20=5y \Rightarrow y=4$
ANSWER: $x = 5, y = 4$

1 (b) (i)

$2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}=4\times 2^{\frac{1}{4}}=2^2\times 2^{\frac{1}{4}}=2^{\frac{9}{4}}$ [Using power rule No. 1: $a^m \times a^n = a^{m+n}$]

1 (b) (ii)

$f(x)=ax^3+bx^2+cx+d$
$f(t)=at^3+bt^2+ct+d$

$f(x)-f(t)=a(x^3-t^3)+b(x^2-t^2)+c(x-t)$ [Factorise using Formulae 1 and 3.]
$\Rightarrow f(x)-f(t)=a(x-t)(x^2+xt+t^2)+b(x-t)(x+t)+c(x-t)$
$\Rightarrow f(x)-f(t)=(x-t)\{a(x^2+xt+t^2)+b(x+t)+c\}$

Therefore, $(x-t)$ is a factor of $f(x)-f(t)$.

1 (c)

Method 1: Division process

$(x-p)^2 = x^2 - 2px + p^2$

$$
\begin{array}{r|l}
 & x + 2p \\
\hline
x^2 - 2px + p^2 & x^3 + 0x^2 + qx + r \\
 & \mp x^3 \pm 2px^2 \mp p^2 x \\
\hline
 & \quad 2px^2 + (q - p^2)x + r \\
 & \quad \mp 2px^2 \pm 4p^2 x \mp 2p^3 \\
\hline
 & \quad (3p^2 + q)x + (r - 2p^3)
\end{array}
$$

The remainder has to be zero, i.e. $0x + 0$.

$\therefore 3p^2 + q = 0 \Rightarrow q = -3p^2$ and $r - 2p^3 = 0 \Rightarrow r = 2p^3$

$27r^2 + 4q^3 = 27(2p^3)^2 + 4(-3p^2)^3 = 27(4p^6) + 4(-27p^6)$

$= 108p^6 - 108p^6 = 0$

$$3x^2 + q = 0 \Rightarrow x = \sqrt{-\frac{q}{3}} = \sqrt{-\frac{-3p^2}{3}} = \sqrt{p^2} = \pm p$$

Method 2: Lining up

$(x-p)^2 = x^2 - 2px + p^2$

$x^3 + 0x^2 + qx + r = (x^2 - 2px + p^2)(x + \frac{r}{p^2})$

$x^3 + 0x^2 + qx + r = x^3 + (\frac{r}{p^2} - 2p)x + (p^2 - \frac{2r}{p})x + r$

Lining up: $(\frac{r}{p^2} - 2p) = 0 \Rightarrow r = 2p^3$ and $(p^2 - \frac{2r}{p}) = q \Rightarrow -3p^2 = q$

$27r^2 + 4q^3 = 27(2p^3)^2 + 4(-3p^2)^3 = 27(4p^6) + 4(-27p^6)$

$= 108p^6 - 108p^6 = 0$

$$3x^2 + q = 0 \Rightarrow x = \sqrt{-\frac{q}{3}} = \sqrt{-\frac{-3p^2}{3}} = \sqrt{p^2} = \pm p$$

2 (a) Solve for x: $|x-1|<7$, where $x \in \mathbf{R}$.

(b) The cubic equation $4x^3+10x^2-7x-3=0$ has one integer root and two irrational roots. Express the irrational roots in simplest surd form.

(c) Let $f(x)=\dfrac{x^2+k^2}{mx}$, where k and m are constants and $m \neq 0$.

(i) Show that $f(km)=f\left(\dfrac{k}{m}\right)$.

(ii) a and b are real numbers such that $a \neq 0$, $b \neq 0$ and $a \neq b$. Show that if $f(a)=f(b)$, then $ab=k^2$.

Solution

1 (a)

Solve the equality: $|x-1|=7 \Rightarrow x-1=\pm 7$

$x-1=7 \Rightarrow x=8$

$x-1=-7 \Rightarrow x=-6$

Do the region test:

←	-6	↔	8	→
-7		0		9
$\|-7-1\|<7$		$\|0-1\|<7$		$\|9-1\|<7$
Wrong		Correct		Wrong

Region Test on $|x-1|<7$ .....**Test Box**

$\therefore -6<x<8$.

1 (b)

The integer root must be a factor of the last term of the cubic expression. It can be any of the following numbers: $-3, -1, 1, 3$.

$f(-3)=4(-3)^3+10(-3)^2-7(-3)-3=-108+90+21-3=0$

Therefore, -3 is a root $\Rightarrow (x+3)$ is a factor.

If you divide (x + 3) into the cubic, you will get the quadratic factor. You can also find the quadratic factor by the lining up process.

Method 1: Division

$$
\begin{array}{r|l}
 & 4x^2-2x-1 \\
\hline
x+3 & 4x^3+10x^2-7x-3 \\
 & \mp 4x^3 \mp 12x^2 \\
\hline
 & -2x^2-7x-3 \\
 & \pm 2x^2 \pm 6x \\
\hline
 & -x-3 \\
 & \pm x \pm 3 \\
\hline
 & 0
\end{array}
$$

Method 2: Lining up

A cubic is a linear multiplied by a quadtatic. You can find the first term and the last term of the quadratic as the first term by the first term gives the first term and the last by the last gives the last. The middle term in the quadratic is unknown so call it *ax*.

$$4x^3+10x^2-7x-3=(x+3)(4x^2+ax-1)$$

$$\Rightarrow 4x^3+10x^2-7x-3=4x^3+(a+12)x^2+(3a-1)x-3$$

Lining up: $a+12=10 \Rightarrow a=-2$

Therefore, the quadratic factor is $4x^2-2x-1$.

Therefore, $4x^3+10x^2-7x-3=(x+3)(4x^2-2x-1)=0$

Solve the quadratic using the formula.

$a=4,\ b=-2,\ c=-1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4+16}}{8}=\frac{2\pm\sqrt{20}}{8}=\frac{2\pm2\sqrt{5}}{8}=\frac{1\pm\sqrt{5}}{4}$$

2 (c) (i)

$$f(km)=\frac{(km)^2+k^2}{m(km)}=\frac{k^2m^2+k^2}{km^2}=\frac{k^2(m^2+1)}{km^2}=\frac{k(m^2+1)}{m^2}$$

$$f\left(\frac{k}{m}\right)=\frac{(\frac{k}{m})^2+k^2}{m(\frac{k}{m})}=\frac{\frac{k^2}{m^2}+k^2}{k}\times\frac{m^2}{m^2}=\frac{k^2+k^2m^2}{km^2}=\frac{k^2(m^2+1)}{km^2}=\frac{k(m^2+1)}{m^2}$$

2 (c) (ii)

$$f(a)=\frac{a^2+k^2}{ma} \text{ and } f(b)=\frac{b^2+k^2}{mb}$$

If $f(a)=f(b)\Rightarrow \dfrac{a^2+k^2}{ma}=\dfrac{b^2+k^2}{mb}\Rightarrow b(a^2+k^2)=a(b^2+k^2)$

$$\Rightarrow ba^2+bk^2=ab^2+ak^2\Rightarrow ba^2-ab^2=ak^2-bk^2$$

$$\Rightarrow ab(a-b)=k^2(a-b)\Rightarrow ab=k^2$$