2004

1 (a) Express $\dfrac{1-\sqrt{3}}{1+\sqrt{3}}$ in the form $a\sqrt{3}-b$, where a and $b \in \mathbf{N}$.

(b) (i) Let $f(x)=x^3+kx^2-4x-12$, where k is a constant. Given that $x+3$ is a factor of $f(x)$, find the value of k.

(ii) Show that $\dfrac{3}{1+x^p}+\dfrac{3}{1+x^{-p}}$ simplifies to a constant.

(c) (i) Show that $p^3+q^3-(p+q)^3=-3pq(p+q)$.

(ii) Hence, or otherwise, find, in terms of a and b, the three values of x for which $(a-x)^3+(b-x)^3-(a+b-2x)^3=0$.

SOLUTION

1 (a)

$$\frac{1-\sqrt{3}}{1+\sqrt{3}}=\frac{(1-\sqrt{3})}{(1+\sqrt{3})}\times\frac{(1-\sqrt{3})}{(1-\sqrt{3})}$$ [Multiply above and below by the conjugate of denominator.]

$$=\frac{1-\sqrt{3}-\sqrt{3}+3}{-2}=\frac{4-2\sqrt{3}}{-2}=\sqrt{3}-2$$

1 (b) (i)

The proof of the factor theorem for a cubic is given in the book. You are asked to prove it for a quadratic, which is easier.

PROOF OF FACTOR THEOREM

$f(x)=ax^2+bx+c$

$f(k)=ak^2+bk+c$

$\therefore\ f(x)-f(k)=a(x^2-k^2)+b(x-k)$

$=a(x+k)(x-k)+b(x-k)$

$=(x-k)\{a(x+k)+b\}=(x-k)g(x)$

$\therefore\ f(x)=f(k)+(x-k)g(x)$

(i) $f(k)=0 \Rightarrow f(x)=(x-k)g(x) \therefore x-k$ is a factor.

(ii) $x-k$ is a factor $\Rightarrow f(k)=0$.

1 (b) (ii)

Always deal with a negative power immediately by multiplying above and below by the opposite positive power.

$$\therefore \frac{3}{1+x^{-p}} \times \frac{x^p}{x^p} = \frac{3x^p}{x^p+1}$$

$$\Rightarrow \frac{3}{1+x^p} + \frac{3}{1+x^{-p}} = \frac{3}{1+x^p} + \frac{3x^p}{1+x^p} = \frac{3+3x^p}{1+x^p} = \frac{3(1+x^p)}{1+x^p} = 3$$

1 (c) (i)

$p^3+q^3-(p+q)^3 = p^3+q^3-(p^3+3p^2q+3pq^2+q^3)$ [Using multiplying out brackets on page 1]

$= p^3+q^3-p^3-3p^2q-3pq^2-q^3 = -3p^2q-3pq^2$

$= -3pq(p-q)$

1 (c) (ii)

$p \leftrightarrow (a-x),\ q \leftrightarrow (b-x),\ p+q \leftrightarrow (a+b-2x)$

If $p^3+q^3-(p+q)^3 = -3pq(p+q)$

$\Rightarrow (a-x)^3+(b-x)^3-(a+b-2x)^3 = -3(a-x)(b-x)(a+b-2x) = 0$

Set each bracket equal to zero to solve for x.

$\Rightarrow x = a,\ b,\ \frac{1}{2}(a+b)$

2 (a) Solve, without using a calculator, the following simultaneous equations:

$$\begin{aligned} 3x+y+z&=0 \\ x-y+z&=2 \\ 2x-3y-z&=9 \end{aligned}$$

(b) (i) Solve the inequality $\dfrac{x+1}{x-1} < 4$, where $x \in \mathbf{R}$ and $x \neq 1$.

(ii) The roots of $x^2+px+q=0$ are α and β, where $p, q \in \mathbf{R}$. Find the quadratic equation whose roots are $\alpha^2\beta$ and $\alpha\beta^2$.

(c) (i) $f(x) = 2x+1$, for $x \in \mathbf{R}$. Show that there exists a real number k such that for all x, $f(x+f(x)) = kf(x)$.

(ii) Show that for any real values of a, b and h, the quadratic equation $(x-a)(x-b)-h^2=0$ has real roots.

SOLUTION

2 (a)

Eliminate z:

$3x+y+z \quad =0...(\mathbf{1})$

$x-y+z \quad =2...(\mathbf{2})$

$2x-3y-z=9...(\mathbf{3})$

Equation **1** + **3** $\Rightarrow 5x-2y=9...(\mathbf{4})$

Equation **2** + **3** $\Rightarrow 3x-4y=11...(\mathbf{5})$

$5x-2y=9...(\mathbf{4})\ (\times -2)$
$3x-4y=11...(\mathbf{5})$

$-10x+4y=-18$
$3x-4y \quad =11$
$-7x \quad =-7 \Rightarrow x=1$

Substituting this value of x into equation **4** $\Rightarrow 5(1)-2y=9 \Rightarrow -2y=4 \Rightarrow y=-2$

Substituting these values of x and y into equation **1** $\Rightarrow 3(1)+(-2)+z=0 \Rightarrow z=-1$

ANSWER: $x=1,\ y=-2,\ z=-1$

2 (b) (i)

Multiply both sides by the denominator squared.

$$\frac{x+1}{x-1}<4 \Rightarrow (x+1)(x-1)<4(x-1)^2 \Rightarrow (x+1)(x-1)-4(x-1)^2<0$$

$(x-1)[(x+1)-4(x-1)]<0$ [Factorise the left-hand side]

$(x-1)[5-3x]<0$

Solve the equality: $(x-1)(5-3x)=0 \Rightarrow x=1,\ \frac{5}{3}$

Do the region test:

$\leftarrow$	1	$\leftrightarrow$	$\frac{5}{3}$	$\rightarrow$
0		$\frac{3}{2}$		2
$(0-1)(5-0)<0$		$(\frac{3}{2}-1)(5-\frac{9}{2})<0$		$(2-1)(5-6)<0$
Correct		Wrong		Correct

Region Test on $(x-1)(5-3x)<0$ .....**Test Box**

$\therefore x<1,\ x>\frac{5}{3}$.

2 (b) (ii)

OLD QUADRATIC

$x^2+px+q=0$

Roots: $\alpha,\ \beta$

Sum **S**: $\alpha+\beta=-p$

Product **P**: $\alpha\beta=q$

NEW QUADRATIC

$x^2-\mathbf{S}x+\mathbf{P}=0$

Roots: $\alpha^2\beta,\ \alpha\beta^2$

Sum **S**: $\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)=-pq$

Product **P**: $(\alpha^2\beta)(\alpha\beta^2)=(\alpha\beta)^3=q^3$

Equation: $x^2+pqx+q^3=0$

2 (c) (i)

$f(x+f(x))=kf(x)\Rightarrow f(x+2x+1)=k(2x+1)$

$\Rightarrow f(3x+1)=k(2x+1)\Rightarrow 2(3x+1)+1=k(2x+1)$

$\Rightarrow 6x+3=k(2x+1)\Rightarrow 3(2x+1)=k(2x+1)$

$\Rightarrow k=3$

2 (c) (ii)

> **Remember**: If $b^2-4ac\geq 0\Rightarrow$ Real roots.
> If $b^2-4ac<0\Rightarrow$ Unreal or complex roots.

$(x-a)(x-b)-h^2=0\Rightarrow x^2-(a+b)x+ab-h^2=0$

If the roots are real you need to show $(a+b)^2-4(ab-h^2)\geq 0.$

$(a+b)^2-4(ab-h^2)=a^2+2ab+b^2-4ab+4h^2=a^2-2ab+b^2+4h^2$

$=(a-b)^2+4h^2\geq 0$ always.