ALGEBRA (Q 1 & 2, PAPER 1)

2003

1 (a) Express the following as a single fraction in its simplest form: $\dfrac{6y}{x(x+4y)}-\dfrac{3}{2x}$.

(b) (i) $f(x)=ax^2+bx+c$ where $a, b, c \in \mathbf{R}$. Given that k is a real number such that $f(k)=0$, prove that $x-k$ is a factor of $f(x)$.

(ii) Show that $2x-\sqrt{3}$ is a factor of $4x^2-2(1+\sqrt{3})x+\sqrt{3}$ and find the other factor.

(c) The real roots of $x^2+10x+c=0$ differ by $2p$ where $c, p \in \mathbf{R}$ and $p>0$.

(i) Show that $p^2=25-c$.

(ii) Given that one root is greater than zero and the other root is less than zero, find the range of possible values of p.

SOLUTION

1 (a)

$$\frac{6y}{x(x+4y)}-\frac{3}{2x}=\frac{12y-3(x+4y)}{2x(x+4y)}=\frac{12y-3x-12y}{2x(x+4y)}$$

$$=\frac{-3x}{2x(x+4y)}=-\frac{3}{2(x+4y)}$$

1 (b) (i)

The proof of the factor theorem for a cubic is given in the book. You are asked to prove it for a quadratic, which is easier.

PROOF OF FACTOR THEOREM

$f(x)=ax^2+bx+c$

$f(k)=ak^2+bk+c$

$\therefore f(x)-f(k)=a(x^2-k^2)+b(x-k)$

$=a(x+k)(x-k)+b(x-k)$

$=(x-k)\{a(x+k)+b\}=(x-k)g(x)$

$\therefore f(x)=f(k)+(x-k)g(x)$

(i) $f(k)=0 \Rightarrow f(x)=(x-k)g(x) \therefore x-k$ is a factor.

(ii) $x-k$ is a factor $\Rightarrow f(k)=0$.

1 (b) (ii)

If $2x-\sqrt{3}$ is a factor of $f(x) \Rightarrow f(\frac{\sqrt{3}}{2})=0$

$\therefore 4(\frac{\sqrt{3}}{2})^2 - 2(1+\sqrt{3})(\frac{\sqrt{3}}{2}) + \sqrt{3} = 3 - \sqrt{3} - 3 + \sqrt{3} = 0$

Therefore, $2x-\sqrt{3}$ is a factor.
You can find the other factor by division or by lining up. Lining up is better here, I think, but you choose whichever option you prefer.
A quadraticis a linear multiplied by a linear. The first term in the quadratic equals the first term in the linear multiplied by the first term in the linear. Also, the last term in the quadratic equals the last term in the linear multiplied by the last term in the linear.

$\therefore 4x^2 - 2(1+\sqrt{3})x + \sqrt{3} = (2x-\sqrt{3})(2x-1)$

$\therefore (2x-1)$ is the other factor.

1 (c) (i)

$x^2 + 10x + c = 0$

Roots: $\alpha,\ \alpha + 2p$ [The roots differ by 2p]

Sum **S**: $\alpha + \alpha + 2p = -10 \Rightarrow \alpha + p = -5$ ... (**1**)

Product **P**: $\alpha(\alpha + 2p) = c$ ... (**2**)

From equation **1**: $\alpha = -p - 5$

Substituting into equation **2**: $(-p-5)(-p-5+2p) = c \Rightarrow (-p-5)(p-5) = c$

$\Rightarrow 25 - p^2 = c \Rightarrow p^2 = 25 - c$

1 (c) (ii)

Call the roots $\alpha,\ \beta$

$\alpha = -p - 5$ [As $p > 0$, this is the negative root]

$\beta = \alpha + 2p = -p - 5 + 2p = p - 5$ [Therefore, this is the positive root. However, it can be seen that it is only positive for values of p greater than 5.]
Answer: $p > 5$

2 (a) Solve the simultaneous equations:

$$3x - y = 8$$
$$x^2 + y^2 = 10$$

(b) (i) Solve for x: $|4x+7| < 1$.

(ii) Given that $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ where $a, b \in \mathbf{R}$, find the value of a and the value of b.

(c) (i) Solve for y: $2^{2y+1} - 5(2^y) + 2 = 0$.

(ii) Given that $x = \alpha$ and $x = \beta$ are the solutions of the quadratic equation $2k^2x^2 + 2ktx + t^2 - 3k^2 = 0$ where $k, t \in \mathbf{R}$ and $k \neq 0$, show that $\alpha^2 + \beta^2$ is independent of k and t.

SOLUTION

2 (a)

Substitute the value of y in the linear equation into the quadratic equation.

$3x - y = 8 \Rightarrow y = 3x - 8$

$x^2 + y^2 = 10 \Rightarrow x^2 + (3x-8)^2 = 10$

$\Rightarrow x^2 + 9x^2 - 48x + 64 - 10 = 0 \Rightarrow 10x^2 - 48x + 54 = 0$

$\Rightarrow 5x^2 - 24x + 27 = 0 \Rightarrow (5x-9)(x-3) = 0$

$\Rightarrow x = \frac{9}{5}, 3$

$x = \frac{9}{5} \Rightarrow y = 3(\frac{9}{5}) - 8 = -\frac{13}{5}$

$x = 3 \Rightarrow y = 3(3) - 8 = 1$

ANSWER: $x = 3, \frac{9}{5};\ y = 1, -\frac{13}{5}$

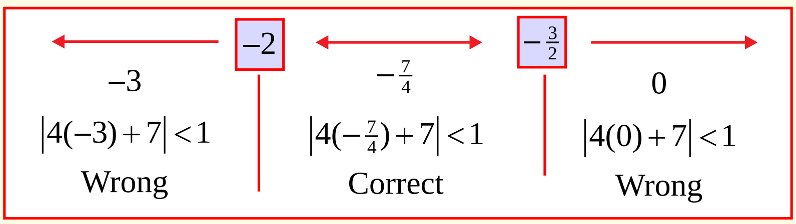
2 (b) (i)

Solve the equality: $|4x+7| = 1 \Rightarrow 4x + 7 = \pm 1$

$x = 1 \Rightarrow 4x + 7 = 1 \Rightarrow 4x = -6 \Rightarrow x = -\frac{3}{2}$

$x = -1 \Rightarrow 4x + 7 = -1 \Rightarrow 4x = -8 \Rightarrow x = -2$

Do the region test:



Region Test on $|4x+7| < 1$ .....**Test Box**

$\therefore -2 < x < -\frac{3}{2}$

2 (b) (ii)

Lining up is the best way to do this.

$x^3 - 5x^2 + bx + 9 = (x^2 - ax - 3)(x - 3)$

$\Rightarrow x^3 - 5x^2 + bx + 9 = x^3 - (a+3)x^2 + (3a-3)x + 9$

Lining up: $a + 3 = 5 \Rightarrow a = 2$ and $3a - 3 = b \Rightarrow b = 3$

2 (c) (i)

Let $u = 2^y$

$2^{2y+1} - 5(2^y) + 2 = 0 \Rightarrow 2(2^y)^2 - 5(2^y) + 2 = 0$

$\Rightarrow 2u^2 - 5u + 2 = 0 \Rightarrow (2u-1)(u-2) = 0 \Rightarrow u = \frac{1}{2},\ 2$

$u = \frac{1}{2} \Rightarrow \frac{1}{2} = 2^y \Rightarrow 2^{-1} = 2^y \Rightarrow y = -1$

$u = 2 \Rightarrow 2 = 2^y \Rightarrow 2^1 = 2^y \Rightarrow y = 1$

Answer: $y = -1, 1$

2 (c) (ii)

$2k^2x^2 + 2ktx + t^2 - 3k^2 = 0$

Roots: $\alpha,\ \beta$

Sum **S**: $\alpha + \beta = -\frac{2kt}{2k^2} = -\frac{t}{k}$

Product **P**: $\alpha\beta = \frac{t^2 - 3k^2}{2k^2}$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{t}{k}\right)^2 - 2\left(\frac{t^2 - 3k^2}{2k^2}\right)$$

$$= \frac{t^2}{k^2} - \frac{t^2 - 3k^2}{k^2} = \frac{3k^2}{k^2} = 3$$