ALGEBRA (Q 1 & 2, PAPER 1)

2002

1 (a) Solve the equation $x = \sqrt{x+2}$.

(b) The cubic equation $x^3 - 4x^2 + 9x - 10 = 0$ has one integer root and two complex roots. Find the three roots.

(c) $(p+r-t)x^2 + 2rx + (t+r-p) = 0$ is a quadratic equation, where *p*, *r*, and *t* are integers. Show that
(i) the roots are rational
(ii) one of the roots is an integer.

SOLUTION

1 (a)

$x = \sqrt{x+2} \Rightarrow x^2 = x+2$ [Squaring both sides.]

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2$

Check both solutions.

$x = -1: -1 = \sqrt{-1+2} \Rightarrow -1 = \sqrt{1}$ [This is incorrect. Therefore, $x = -1$ is not a solution.]

$x = 2:\ 2 = \sqrt{2+2} \Rightarrow 2 = \sqrt{4}$ [This is correct. Therefore $x = 2$ is a solution.]

ANSWER: $x = 2$

1 (b)

The integer root must be a factor of the last term of the cubic expression. It can be any of the following numbers: $-10, -5, -1, 1, 5, 10$.

$f(2) = (2)^3 - 4(2)^2 + 9(2) - 10 = 8 - 16 + 18 - 10 = 0$

Therefore, 2 is a root $\Rightarrow (x-2)$ is a factor.

If you divide $(x-2)$ into the cubic, you will get the quadratic factor. You can also find the quadratic factor by the lining up process.

METHOD 1: DIVISION

$$
\begin{array}{r|l}
 & x^2 - 2x + 5 \\
\hline
(x-2) & x^3 - 4x^2 + 9x - 10 \\
 & \mp x^3 \pm 2x^2 \\
\hline
 & -2x^2 + 9x - 10 \\
 & \pm 2x^2 \mp 4x \\
\hline
 & 5x - 10 \\
 & \mp 5x \pm 10 \\
\hline
 & 0
\end{array}
$$

METHOD 2: LINING UP
A cubic is a linear multiplied by a quadtatic. You can find the first term and the last term of the quadratic as the first term by the first term gives the first term and the last by the last gives the last. The middle term in the quadratic is unknown so call it *ax*.

$x^3 - 4x^2 + 9x - 10 = (x-2)(x^2 + ax + 5)$

$\Rightarrow x^3 - 4x^2 + 9x - 10 = x^3 + (a-2)x^2 + (5-2a)x - 10$

Lining up: $a - 2 = -4 \Rightarrow a = -2$

Therefore, the quadratic factor is $x^2 - 2x + 5$.

$\therefore x^3 - 4x^2 + 9x - 10 = (x-2)(x^2 - 2x + 5)$

Solve the quadratic using the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

ANSWER: $2,\ 1 \pm 2i$

1 (c)

$(p + r - t)x^2 + 2rx + (t + r - p) = 0$

$a = (p + r - t) = [r + (p - t)]$

$b = 2r$

$c = (t + r - p) = [r - (p - t)]$

$$x = \frac{-2r \pm \sqrt{4r^2 - 4[r + (p-t)][r - (p-t)]}}{2(p + r - t)}$$

$$= \frac{-2r \pm \sqrt{4r^2 - 4r^2 + 4(p-t)^2}}{2(p + r - t)} = \frac{-2r \pm \sqrt{4(p-t)^2}}{2(p + r - t)} = \frac{-2r \pm 2(p-t)}{2(p + r - t)}$$

$$x = \frac{-2r + 2p - 2t}{2(p + r - t)} = \frac{2(p - r - t)}{2(p + r - t)} = \frac{p - r - t}{p + r - t}$$

or $x = \dfrac{-2r - 2p + 2t}{2(p + r - t)} = \dfrac{-2(p + r - t)}{2(p + r - t)} = -1$

2 (a) Solve, without using a calculator, the following simultaneous equations:

$$x+2y+4z=7$$
$$x+3y+2z=1$$
$$-y+3z=8$$

(b) (i) Find the range of values of $x \in \mathbf{R}$ for which $x^2+x-20 \leq 0$.

(ii) Let $g(x)=x^n+3$, for all $x \in \mathbf{R}$, where $n \in \mathbf{N}$. Show that if n is odd then $g(x)+g(-x)$ is constant.

(c) (i) Show that if the roots of $x^2+bx+c=0$ differ by 1, then $b^2-4c=1$.

(ii) The roots of the equation $x^2+(4k-5)x+k=0$ are consecutive integers. Using the result from part (i), or otherwise, find the value of k and the roots of the equation.

SOLUTION

2 (a)

Eliminate *x* form equations **1** and **2**:

$$x+2y+4z=7 \ldots(\mathbf{1})$$
$$x+3y+2z=1 \ldots(\mathbf{2})$$
$$-y+3z=8 \ldots(\mathbf{3})$$

$$x+2y+4z=7 \ldots(\mathbf{1})$$
$$x+3y+2z=1 \ldots(\mathbf{2})\,(\times -1)$$

$$x+2y+4z=7$$
$$-x-3y-2z=-1$$
$$-y+2z=6 \ldots(\mathbf{4})$$

Now combine equations **3** and **4** to eliminate *y*:

$$-y+3z=8 \ldots(\mathbf{3})$$
$$-y+2z=6 \ldots(\mathbf{4})\,(\times -1)$$

$$-y+3z=8$$
$$y-2z=-6$$
$$z=2$$

Substituting this value of *z* into equation **3**

$\Rightarrow -y+3(2)=8 \Rightarrow y=-2$

Substituting these values of *y* and *z* into equation **1**

$\Rightarrow x+2(-2)+4(2)=7 \Rightarrow x=3$

ANSWER: $x=3,\ y=-2,\ z=2$

2 (b) (i)

Solve the equality: $x^2+x-20=0 \Rightarrow (x+5)(x-4)=0 \Rightarrow x=-5,\ 4$

Do the region test:

←	-5	↔	4	→
-6		0		5
$(-6)^2+(-6)-20 \leq 0$		$(0)^2+(0)-20 \leq 0$		$(5)^2+(5)-20 \leq 0$
Wrong		Correct		Wrong

Region Test on $x^2+x-20 \leq 0$ .....**Test Box**

$\therefore -5 \leq x \leq 4.$

2 (b) (ii)

A negative number raised to an even power gives a positive answer. However, a negative number raised to an odd power gives a negative answer.

$g(x) = x^n + 3$

$g(-x) = (-x)^n + 3 = -x^n + 3,$ as n is an odd power.

$\therefore g(x) + g(-x) = x^n + 3 - x^n + 3 = 6$ [i.e. a constant]

2 (c) (i)

$x^2 + bx + c = 0$

Roots: $\alpha,\ \alpha + 1$

Sum **S**: $\alpha + \alpha + 1 = -b \Rightarrow 2\alpha + 1 = -b$...(**1**)

Product **P**: $\alpha(\alpha + 1) = c$...(**2**)

From equation **1**: $2\alpha + 1 = -b \Rightarrow \alpha = -\dfrac{b+1}{2}$

Substituting into equation **2**:

$$\alpha(\alpha + 1) = c \Rightarrow \left(-\frac{b+1}{2}\right)\left(-\frac{b+1}{2} + 1\right) = c \Rightarrow \left(\frac{-b-1}{2}\right)\left(\frac{-b+1}{2}\right) = c$$

$\Rightarrow b^2 - 1 = 4c \Rightarrow b^2 - 4c = 1$

2 (c) (ii)

$x^2 + (4k - 5)x + k = 0$

As the roots are consecutive integers $\Rightarrow b^2 - 4c = 1$.

$\therefore (4k-5)^2 - 4k = 1 \Rightarrow 16k^2 - 40k + 25 - 4k - 1 = 0$

$\Rightarrow 16k^2 - 44k + 24 = 0 \Rightarrow 4k^2 - 11k + 6 = 0$

$\Rightarrow (4k-3)(k-2) = 0 \Rightarrow k = \frac{3}{4},\ 2$

Which value of k gives integer roots?

$k = \frac{3}{4} \Rightarrow x^2 + (4(\frac{3}{4}) - 5)x + (\frac{3}{4}) = 0 \Rightarrow x^2 - 2x + \frac{3}{4} = 0$

$\Rightarrow 4x^2 - 8x + 3 = 0 \Rightarrow (2x-3)(2x-1) = 0 \Rightarrow x = \frac{1}{2},\ \frac{3}{2}$ [Not integer roots]

$k = 2 \Rightarrow x^2 + (4(2) - 5)x + 2 = 0 \Rightarrow x^2 + 3x + 2 = 0$

$\Rightarrow (x+2)(x+1) = 0 \Rightarrow x = -2,\ -1$ [Integer roots]

Answer: $k = 2$; Roots: $-2,\ -1$