ALGEBRA (Q 1 & 2, PAPER 1)

2001

1 (a) Find the real numbers a and b such that $x^2+4x-6=(x+a)^2+b$ for all $x\in\mathbf{R}$.

(b) Let $f(x)=2x^3+mx^2+nx+2$ where m and n are constants. Given that $x-1$ and $x+2$ are factors of $f(x)$, find the value of m and the value of n.

(c) x^2-px+q is a factor of $x^3+3px^2+3qx+r$.

(i) Show that $q=-2p^2$.

(ii) Show that $r=-8p^3$.

(iii) Find the three roots of $x^3+3px^2+3qx+r=0$ in terms of p.

SOLUTION

1 (a)

Multiply out the brackets and line up the coefficients.

$$x^2+4x-6=(x+a)^2+b \Rightarrow x^2+4x-6=x^2+2ax+a^2+b$$

Therefore, $2a=4$ and $a^2+b=-6 \Rightarrow a=2,\ b=-10$

1 (b)

$x-1$ is a factor $\Rightarrow f(1)=2(1)^3+m(1)^2+n(1)+2=0 \Rightarrow m+n=-4$.....(**1**)

$x+2$ is a factor $\Rightarrow f(-2)=2(-2)^3+m(-2)^2+n(-2)+2=0 \Rightarrow 2m-n=7$.....(**2**)

Solving equation **1** and **2** simultaneously: $m=1,\ n=-5$

1 (c)

The division method only is shown here. Try lining up yourself if that is your favoured method.

$$\begin{array}{r|l} & x+4p \\ x^2-px+q & x^3+3px^2+3qx+r \\ & \mp x^3 \pm px^2 \mp qx \\ \hline & 4px^2+2qx+r \\ & \mp 4px^2 \pm 4p^2x \mp 4pq \\ \hline & (4p^2+2q)x+(r-4pq) \end{array}$$

The remainder has to be zero, i.e. $0x + 0$.

$$\therefore 4p^2+2q=0 \Rightarrow q=-2p^2$$

and $r=4pq \Rightarrow r=4p(-2p^2)=-8p^3$

$$x^3+3px^2+3qx+r=(x^2-px+q)(x+4p)$$

$$\Rightarrow x^3+3px^2+3qx+r=(x^2-px-2p^2)(x+4p)=(x-2p)(x+p)(x+4p)$$

$$\therefore x=-4p,\ -p,\ 2p$$

2 (a) Solve the simultaneous equations:

$$x - y = 0$$
$$(x+2)^2 + y^2 = 10$$

(b) (i) Solve for x: $|3x+5| < 4$.

(ii) Simplify $\left(x^2 + \sqrt{2} + \frac{1}{x^2}\right)\left(x^2 - \sqrt{2} + \frac{1}{x^2}\right)$ and express your answer in the form

$x^n + \frac{1}{x^n}$ where n is a whole number.

(c) α and β are real numbers such that $\alpha + \beta = -7$ and $\alpha\beta = 11$.

(i) Show that $\alpha^2 + \beta^2 = 27$.

(ii) Find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ and write your answer in the form

$px^2 + qx + r = 0$ where $p, q, r \in \mathbf{Z}$.

SOLUTION

2 (a)

From the linear equation $x = y$.

Substituting for y in the quadratic $\Rightarrow (x+2)^2 + x^2 = 10 \Rightarrow x^2 + 4x + 4 + x^2 - 10 = 0$

$\Rightarrow 2x^2 + 4x - 6 = 0 \Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0 \Rightarrow x = -3, 1 \Rightarrow y = -3, 1$

ANSWER: $x = -3, 1;\ y = -3, 1$

2 (b) (i)

Solve the equality: $|3x+5| = 4 \Rightarrow 3x + 5 = \pm 4$

$3x + 5 = 4 \Rightarrow 3x = -1 \Rightarrow x = -\frac{1}{3}$

$3x + 5 = -4 \Rightarrow 3x = -9 \Rightarrow x = -3$

Do the region test:

$x < -3$	-3	$-3 < x < -\frac{1}{3}$	$-\frac{1}{3}$	$x > -\frac{1}{3}$
-4		-1		0
$\lvert 3(-4)+5 \rvert < 4$		$\lvert 3(-1)+5 \rvert < 4$		$\lvert 3(0)+5 \rvert < 4$
Wrong		Correct		Wrong

Region Test on $|3x+5| < 4$ .....**Test Box**

$\therefore -3 < x < -\frac{1}{3}$

2 (b) (ii)

$$\left(x^2+\sqrt{2}+\frac{1}{x^2}\right)\left(x^2-\sqrt{2}+\frac{1}{x^2}\right)=x^4-\sqrt{2}x^2+1+\sqrt{2}x^2-2+\frac{\sqrt{2}}{x^2}+1-\frac{\sqrt{2}}{x^2}+\frac{1}{x^4}$$

$$=x^4+\frac{1}{x^4}$$

2 (c) (i)

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-7)^2-2(11)=49-22=27$$

2 (c) (ii)

Roots: $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Sum **S**: $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{27}{11}$

Product **P**: $\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)=1$

Using $x^2-\mathbf{S}x+\mathbf{P}=0 \Rightarrow x^2-\frac{27}{11}x+1=0 \Rightarrow 11x^2-27x+11=0$