Sequences & Series (Q 4 & 5, Paper 1)

Lesson No. 8: Some Extra Algebra

2006

5 (c) (i) Given two real numbers a and b, where $a > 1$ and $b > 1$, prove that

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2.$$

(ii) Under what condition is $\frac{1}{\log_b a} + \frac{1}{\log_a b} = 2$.

Solution

5 (c) (i)

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2 \Rightarrow \log_a b + \frac{1}{\log_a b} \geq 2 \text{ [Multiply across by } \log_a b\text{]}$$

$$\Rightarrow (\log_a b)^2 + 1 \geq 2\log_a b \Rightarrow (\log_a b)^2 - 2\log_a b + 1 \geq 0$$

$$\Rightarrow (\log_a b - 1)(\log_a b - 1) \geq 0 \Rightarrow (\log_a b - 1)^2 \geq 0 \text{ [This is always true]}$$

5 (c) (ii)

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} = 2 \Rightarrow \log_a b + \frac{1}{\log_a b} = 2$$

Let $u = \log_a b \Rightarrow u + \frac{1}{u} = 2 \Rightarrow u^2 - 2u + 1 = 0$

$$\Rightarrow (u-1)(u-1) = 0 \Rightarrow u = 1$$

$$\Rightarrow \log_a b = 1 \Rightarrow b = a$$

2005

5 (a) Solve for x: $\sqrt{10 - x} = 4 - x$.

Solution

$\sqrt{10-x} = 4 - x \Rightarrow 10 - x = (4-x)^2$ [Squaring both sides.]

$$\Rightarrow 10 - x = 16 - 8x + x^2 \Rightarrow x^2 - 7x + 6 = 0$$

$$\Rightarrow (x-6)(x-1) = 0 \Rightarrow x = 6, 1$$

Check both solutions:

$x = 6$: $\sqrt{10-6} = 4 - 6 \Rightarrow \sqrt{4} = -2$ [Wrong]

$x = 1$: $\sqrt{10-1} = 4 - 1 \Rightarrow \sqrt{9} = 3$ [Correct]

2003

5 (a) Solve for *x*: $x=\sqrt{7x-6}+2$.

SOLUTION

$x=\sqrt{7x-6}+2 \Rightarrow (x-2)=\sqrt{7x-6}$ [Isolate the surd expression.]

$\Rightarrow (x-2)^2=7x-6 \Rightarrow x^2-4x+4=7x-6$

$\Rightarrow x^2-11x+10=0 \Rightarrow (x-10)(x-1)=0 \Rightarrow x=1, 10$

Check solutions:

x = 1: $1=\sqrt{7(1)-6}+2 \Rightarrow 1=\sqrt{1}+2 \Rightarrow 1=1+2$ [Not a solution]

x = 10: $10=\sqrt{7(10)-6}+2 \Rightarrow 10=\sqrt{64}+2 \Rightarrow 10=8+2$ [Works]

ANSWER: x = 10

2005

5 (c) (i) Show that $\dfrac{1}{\log_a b}=\log_b a$, where $a, b>0$ and $a, b\neq 1$.

(ii) Show that $\dfrac{1}{\log_2 c}+\dfrac{1}{\log_3 c}+\dfrac{1}{\log_4 c}+.......+\dfrac{1}{\log_r c}=\dfrac{1}{\log_{r!} c}$, where $c>0$, $c\neq 1$.

SOLUTION

5 (c) (i)

$$\log_b a=\frac{\log_a a}{\log_a b}=\frac{1}{\log_a b}$$

LOG RULES

$\log_a M=\dfrac{\log_b M}{\log_b a}$ [Used to change base]

$\log_a a=1$

5 (c) (ii)

$$\frac{1}{\log_2 c}+\frac{1}{\log_3 c}+\frac{1}{\log_4 c}+.......+\frac{1}{\log_r c}=\log_c 2+\log_c 3+\log_c 4+.......+\log_c r$$

$$=\log_c(2\times3\times4\times....\times r)=\log_c(r!)=\frac{1}{\log_{r!} c}$$

2004

5 (b) (ii) Solve $\log_4 x-\log_4(x-2)=\tfrac{1}{2}$.

SOLUTION

$$\log_4 x-\log_4(x-2)=\tfrac{1}{2}\Rightarrow \log_4\left(\frac{x}{x-2}\right)=\tfrac{1}{2}$$

$$\Rightarrow\left(\frac{x}{x-2}\right)=4^{\frac{1}{2}}=2\Rightarrow x=2(x-2)\Rightarrow x=2x-4\Rightarrow x=4$$

LOG RULES

2. $\log_a M-\log_a N=\log_a\left(\dfrac{M}{N}\right)$

2002

5 (a) Find the value of x in each case:

(i) $\frac{8}{2^x} = 32$

(ii) $\log_9 x = \frac{3}{2}$.

Solution

5 (a) (i)

$$\frac{8}{2^x} = 32 \Rightarrow \frac{2^3}{2^x} = 2^5$$

$$\Rightarrow 2^{3-x} = 2^5 \Rightarrow 3 - x = 5 \Rightarrow x = -2$$

5 (a) (ii)

$$\log_9 x = \frac{3}{2} \Rightarrow x = 9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$$

2001

5 (b) (i) Solve $\log_6(x+5) = 2 - \log_6 x$ for $x > 0$.

Solution

$$\log_6(x+5) = 2 - \log_6 x \Rightarrow \log_6(x+5) + \log_6 x = 2$$

$$\Rightarrow \log_6 x(x+5) = 2 \Rightarrow x(x+5) = 6^2 = 36$$

$$\Rightarrow x^2 + 5x - 36 = 0 \Rightarrow (x-4)(x+9) = 0 \Rightarrow x = 4, -9$$

Check both solution. Only $x = 4$ works. $x = -9$ gives negative logs which are not allowed.