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SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

LESSON NO. 7: PROOFS BY INDUCTION

2005

5 (b) Prove by induction that $\sum_{r=1}^{n}(3r-2)=\frac{n}{2}(3n-1)$.

SOLUTION

STEPS

1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.

2. Assuming result is true for $n = k$.

3. Prove result is true for $n = (k+1)$.

SUMMATIONS: When proving a summation always expand as follows:

1. Prove it is true for $n = 1$: $\sum_{r=1}^{1}(3r-2)=3(1)-2=1$ and $\frac{n}{2}(3n-1)=\frac{1}{2}(3(1)-1)=1$

2. Assume it is true for $n = k$: $\sum_{r=1}^{k}(3r-2)=\{1+4+7+......+3k-2\}=\frac{k}{2}(3k-1)$

3. Prove it is true for $n = k + 1$:

You need to prove that $\sum_{r=1}^{k+1}(3r-2)=\{1+4+7+......+3k-2\}+(3k+1)=\frac{k+1}{2}(3k+2)$

Using the result in step 2 $\Rightarrow \frac{k}{2}(3k-1)+(3k+1)=\frac{k+1}{2}(3k+2)$

$\Rightarrow k(3k-1)+2(3k+1)=(k+1)(3k+2)$

$\Rightarrow 3k^2-k+6k+2=(k+1)(3k+2) \Rightarrow 3k^2+5k+2=(k+1)(3k+2)$

$\Rightarrow (k+1)(3k+2)=(k+1)(3k+2)$

2002

5 (c) Prove by induction that, for any positive integer n, $x+x^2+x^3+\ldots+x^n=\dfrac{x(x^n-1)}{x-1}$,

where $x \neq 1$.

SOLUTION

STEPS

1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.

2. Assuming result is true for n = k.

3. Prove result is true for n = (k +1).

Rewrite as: $\displaystyle\sum_{r=1}^{n} x^r = \frac{x(x^n-1)}{x-1}$

1. Prove for n = 1: $\displaystyle\sum_{r=1}^{1} x^r = x^1 = x$ and $\dfrac{x(x^n-1)}{x-1}=\dfrac{x(x-1)}{x-1}=x$ [True for n = 1]

2. Assume for n = k: $\displaystyle\sum_{r=1}^{k} x^r = \{x+x^2+x^3+\ldots+x^k\} = \frac{x(x^k-1)}{x-1}$

3. Prove for n = k +1: $\displaystyle\sum_{r=1}^{k+1} x^r = \{x+x^2+x^3+\ldots+x^k\}+x^{k+1} = \frac{x(x^{k+1}-1)}{x-1}$

Using step 2: $\Rightarrow \dfrac{x(x^k-1)}{x-1}+x^{k+1}=\dfrac{x(x^{k+1}-1)}{x-1} \Rightarrow \dfrac{x(x^k-1)x^{k+1}(x-1)}{x-1}=\dfrac{x(x^{k+1}-1)}{x-1}$

$\Rightarrow \dfrac{x(x^k-1)+x^{k+1}(x-1)}{x-1}=\dfrac{x(x^{k+1}-1)}{x-1}$

$\Rightarrow \dfrac{x^{k+1}-x+x^{k+2}-x^{k+1}}{x-1}=\dfrac{x(x^{k+1}-1)}{x-1} \Rightarrow \dfrac{x^{k+2}-x}{x-1}=\dfrac{x(x^{k+1}-1)}{x-1} \Rightarrow \dfrac{x(x^{k+1}-1)}{x-1}=\dfrac{x(x^{k+1}-1)}{x-1}$

Therefore, true for n = k + 1.

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2003

5 (b) Use induction to prove that 8 is a factor of $7^{2n+1}+1$ for any positive integer *n*.

SOLUTION

STEPS
1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
2. Assuming result is true for $n = k$.
3. Prove result is true for $n = (k+1)$.

1. Prove for $n = 1$.

$7^{2(1)+1}+1=7^3+1=343+1=344$

$344 \div 8 = 43$ [Therefore, true for $n = 1$.]

2. Assume for $n = k \Rightarrow 7^{2k+1}+1=8m,\ m \in \mathbf{N}_0$.

3. Prove for $n = k + 1$.

$\Rightarrow 7^{2(k+1)+1}+1=7^{2k+3}+1=7^2(7^{2k+1})+1=49(7^{2k+1})+1$

From step 2: $7^{2k+1}=8m-1$

$\Rightarrow 7^{2k+3}+1=49(8m-1)+1=49(8m)+48=8(49m+6)=8a,\ a \in \mathbf{N}_0$.

2004

5 (c) Prove by induction that $2^n \geq n^2,\ n \in \mathbf{N},\ n \geq 4$.

SOLUTION

STEPS
1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
2. Assuming result is true for $n = k$.
3. Prove result is true for $n = (k+1)$.

Prove $2^n \geq n^2$ for all $n \geq 4$.

Rewrite it as: Prove $n^2 \leq 2^n$ for all $n \geq 4$.

1. Prove this statement is true for $n = 4$.

2. Assume it is true for $n = k \Rightarrow k^2 \leq 2^k$

3. Prove for $n = k + 1$. Show that $\Rightarrow (k+1)^2 \leq 2^{k+1} \Rightarrow k^2(1+\frac{1}{k})^2 \leq 2^k \times 2$

From Step 2: $k^2 \leq 2^k$

$k \geq 4 \Rightarrow \frac{1}{k} \leq \frac{1}{4} \Rightarrow 1+\frac{1}{k} \leq \frac{5}{4}$

$\Rightarrow (1+\frac{1}{k})^2 \leq \frac{25}{16} \leq 2$

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2001

5 (c) Use induction to prove that, for n a positive integer, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

for all $\theta \in \mathbf{R}$ and $i^2 = -1$.

Solution

Statement of De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbf{N}_0$.

Proof

1. For $n = 1$: Prove $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

i.e. $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$. This is obviously true.

2. For $n = k$: Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

3. For $n = k + 1$: Prove $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

Proof: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ using **Step 2**

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.