SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

LESSON NO. 3: GEOMETRIC SEQUENCES

2006

4 (b) The sum to infinity of a geometric series is $\frac{9}{2}$. The second term of the series is -2. Find the vaue of r, the common ratio of the series.

SOLUTION

$S_\infty = \frac{9}{2}$, $ar = -2$

$S_\infty = \frac{a}{1-r}, -1 < r < 1$ 6

$$S_\infty = \frac{a}{1-r} = \frac{9}{2} \Rightarrow 2a = 9 - 9r \text{ ....(1)}$$

$ar = -2 \Rightarrow a = -\frac{2}{r}$ ....(**2**)

Substituting equation (**2**) into equation (**1**):

$$-\tfrac{4}{r} = 9 - 9r \Rightarrow -4 = 9r - 9r^2 \Rightarrow 9r^2 - 9r - 4 = 0$$

$$\Rightarrow (3r-4)(3r+1) = 0 \Rightarrow r = -\tfrac{1}{3}, \tfrac{4}{3}$$

$r = -\frac{1}{3}$ is the only solution as the sum to infinity formula only applies to values of r between -1 and 1.

4 (c) The sequence $u_1, u_2, u_3, \ldots,$ defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3,$ is as follows:
3, 9, 21, 45, 93, ...

(i) Find u_6, and verify that it is equal to the sum of the first six terms of a geometric series with first term 3 and common ratio 2.

(ii) Given that, for all k, u_k is the sum of the first k terms of a geometric series with first term 3 and common ratio 2, find $\sum_{k=1}^{n} u_k$.

SOLUTION

4 (c) (i)

$u_{n+1} = 2u_n + 3 \Rightarrow u_6 = 2u_5 + 3 = 2(93) + 5 = 189$

Summing formula: $S_n = \frac{a(1-r^n)}{(1-r)}$ 5

Geometric Series: $a = 3$, $r = 2$, $n = 6$

$$S_6 = \frac{3(1-2^6)}{1-2} = -3(1-64) = -3(-63) = 189$$

CONT...

4 (c) (ii)

$$u_k = \frac{3(1-2^k)}{1-2} = 3(2^k - 1) = 3(2^k) - 3$$

To evaluate $\sum_{k=1}^{n} u_k$ set out a sum table.

SUM TABLE

$k = 1$: $3(2) - 3$

$k = 2$: $3(2^2) - 3$

$k = 3$: $3(2^3) - 3$

⋮

$k = n-1$: $3(2^{n-1}) - 3$

$k = n$: $3(2^n) - 3$

$$\therefore \sum_{k=1}^{n} u_k = 3(2 + 2^2 + 2^3 + ... + 2^n) - 3n$$

Inside the bracket, you have a geometric series with a = 2 and r = 2.

$$\Rightarrow \sum_{k=1}^{n} u_k = 3\left(\frac{2(1-2^n)}{1-2}\right) - 3n$$

$$\Rightarrow \sum_{k=1}^{n} u_k = 6(2^n - 1) - 3n$$

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ **5**

2004

5 (b) (i) In a geometric series, the second term is 8 and the fifth term is 27. Find the first term and the common ratio.

SOLUTION

$$\frac{ar^4 = 27}{ar = 8} \Rightarrow r^3 = \tfrac{27}{8} \Rightarrow r = \tfrac{3}{2}$$

$$ar = 8 \Rightarrow a(\tfrac{3}{2}) = 8 \Rightarrow a = \tfrac{16}{3}$$

> The forty-third term of a geometric sequence is written as $u_{43} = ar^{42}$

2002

4 (a) Find in terms of n, the sum of the first n terms of the geometric series $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + ...$

SOLUTION

4 (a)

$a = 3$, $r = \frac{1}{2}$

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ **5**

$$\therefore S_n = \frac{3(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = 6(1-(\tfrac{1}{2})^n)$$

2001

5 (a) The second term, u_2, of a geometric sequence is 21. The third term, u_3, is -63. Find

(i) the common ratio

(ii) the first term.

SOLUTION

5 (a) (i)

$$\frac{u_3 = ar^2 = -63}{u_2 = ar = 21} \quad \text{Dividing} \Rightarrow r = -3$$

The forty-third term of a geometric sequence is written as $u_{43} = ar^{42}$

5 (a) (ii)

$ar = 21 \Rightarrow a = \frac{21}{r} = \frac{21}{-3} = -7$

2005

4 (a) Write the recurring decimal 0·636363.... as an infinite geometric series and hence as a fraction.

SOLUTION

4 (a)

$0\cdot636363... = \frac{63}{100} + \frac{63}{10000} + \frac{63}{1000000} + ... = 63(\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + ...)$

Infinite geometric series: $a = \frac{1}{100}$, $r = \frac{1}{100}$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99}{100}} = \frac{1}{99}$$

$\therefore 0\cdot636363.... = 63(\frac{1}{99}) = \frac{63}{99}$

2003

4 (a) Express the recurring decimal 0.252525... in the form $\frac{p}{q}$ where $p, q \in \mathbf{N}$ and $q \neq 0$.

SOLUTION

4 (a)

$0\cdot252525... = \frac{25}{100} + \frac{25}{10000} + \frac{25}{1000000} + ... = 25(\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + ...)$

Infinite geometric series: $a = \frac{1}{100}$, $r = \frac{1}{100}$

$S_\infty = \frac{a}{1-r}, -1 < r < 1$ 6

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99}{100}} = \frac{1}{99}$$

$\therefore 0\cdot252525.... = 25(\frac{1}{99}) = \frac{25}{99}$