SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2007

4 (a) Show that $\binom{n}{1}+\binom{n}{2}=\binom{n+1}{2}$ for all natural numbers $n \geq 2$.

(b) $u_1 = 5$ and $u_{n+1} = \frac{n}{n+1}u_n$ for all $n \geq 1, n \in \mathbf{N}$.

(i) Write down the value of each of u_2, u_3, and u_4.

(ii) Hence, by inspection, write an expression for u_n in terms of n.

(iii) Use induction to justify your answer for part **(ii)**.

(c) The sum of the first n terms of a series is given by $S_n = n^2 \log_e 3$.

(i) Find the n^{th} term and prove that the series is arithmetic.

(ii) How many of the terms of the series are less than $12\log_e 27$?

SOLUTION

4 (a)

LHS

$$\binom{n}{1}+\binom{n}{2}$$

$$= n + \frac{n(n-1)}{2} = \frac{2n+n^2-n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

RHS

$$\binom{n+1}{2} = \frac{(n+1)n}{2}$$

4 (b) (i)

Put n = 1: $u_2 = \frac{1}{2}u_1 = \frac{1}{2}\times 5 = \frac{5}{2}$

Put n = 2: $u_3 = \frac{2}{3}u_2 = \frac{2}{3}\times \frac{5}{2} = \frac{5}{3}$

Put n = 3: $u_4 = \frac{3}{4}u_3 = \frac{3}{4}\times \frac{5}{3} = \frac{5}{4}$

4 (b) (ii)

Sequence: $5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \ldots\ldots, \frac{5}{n}$

By inspection, it is seen that $u_n = \frac{5}{n}$

4 (b) (iii)

Steps
1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
2. Assuming result is true for $n = k$.
3. Prove result is true for $n = (k+1)$.

The sequence is $5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \ldots\ldots, \frac{5}{n}$

The general term is assumed to be $u_n = \frac{5}{n}$ by inspection.

1. Prove for $n = 1$: $u_1 = \frac{5}{1} = 5$ (True for $n = 1$)

2. Assume for $n = k$: $u_k = \frac{5}{k}$

3. Prove for $n = k + 1$: $u_{k+1} = \dfrac{5}{k+1}$

From step **2**: $5 = ku_k \Rightarrow u_{k+1} = \dfrac{ku_k}{k+1} = \dfrac{k}{k+1} \times \dfrac{5}{k} = \dfrac{5}{k+1}$ (True for $n = k + 1$)

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.
So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.

4 (c) (i)

$S_n = n^2 \ln 3$

$S_{n-1} = (n-1)^2 \ln 3$

$u_n = S_n - S_{n-1}$ 1

$\therefore u_n = S_n - S_{n-1} = n^2 \ln 3 - (n-1)^2 \ln 3$

$= n^2 \ln 3 - n^2 \ln 3 + 2n \ln 3 - \ln 3$

$\therefore u_n = (2n-1)\ln 3$

Test for an arithmetic sequence: $u_{n+1} - u_n = \text{Constant} = d$

$\therefore u_{n+1} - u_n = (2n+1)\ln 3 - (2n-1)\ln 3 = 2n \ln 3 + \ln 3 - 2n \ln 3 + \ln 3$

$= 2 \ln 3$ (Constant)

4 (c) (ii)

Set the general term equal to $12 \ln 27$.

Use to change base:
$\log_a M = \dfrac{\log_b M}{\log_b a}$

$\therefore (2n-1)\ln 3 = 12 \ln 27 \Rightarrow (2n-1) = \dfrac{12 \ln 27}{\ln 3} = 12 \dfrac{\log_e 27}{\log_e 3}$

$\Rightarrow (2n-1) = 12 \log_3 27 = 12 \times 3 = 36$

$\Rightarrow 2n = 37 \Rightarrow n = 18 \cdot 5$

Therefore, there are 18 terms less than $12 \ln 27$.

5 (a) Plot, on the number line, the values of x that satisfy the inequality
$|x+1| \le 2$, where $x \in \mathbf{Z}$.

(b) In the expansion of $\left(2x - \frac{1}{x^2}\right)^9$,

(i) find the general term

(ii) find the value of the term independent of x.

(c) The n^{th} term of a series is given by nx^n, where $|x| < 1$.

(i) Find an expression for S_n, the sum of the first n terms of the series.

(ii) Hence, find the sum to infinity of the series.

SOLUTION

5 (a)

MODULUS $|ax+b| > c$

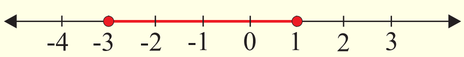
STEPS
1. Solve the corresponding modulus equality.
2. Do region test on roots in ascending order on **Test Box**.
3. Based on the region test write down the solutions.

1. Solve $|x+1| = 2 \Rightarrow x+1 = \pm 2 \Rightarrow x = -3, 1$

2. Region Test on $|x+1| \le 2$ .....**Test Box**

←	-3	↔	1	→
-4		0		2
$\lvert -3 \rvert \le 2$		$\lvert 1 \rvert \le 2$		$\lvert 3 \rvert \le 2$
Wrong		Correct		Wrong

3. $\therefore -3 \le x \le 1$



5 (b) (i)

$$u_{r+1} = \binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x}\right)^r$$

$$u_{r+1} = {}^nC_r\,(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r$$ **10**

5 (b) (ii)

$$u_{r+1} = \binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x^2}\right)^r \Rightarrow u_{r+1} = (-1)^r\binom{9}{r}2^{9-r}\frac{x^{9-r}}{x^{2r}} = (-1)^r\binom{9}{r}2^{9-r}x^{9-3r}$$

Independent term: $9 - 3r = 0 \Rightarrow r = 3$

$$\therefore u_4 = (-1)^3\binom{9}{3}2^6\frac{x^6}{x^6} = (-1)\times 84 \times 64 = -5376$$

5 (c) (i)

$u_n = nx^n \Rightarrow S_n = x + 2x^2 + 3x^2 + ... + nx^n$

$S_n \ = x + 2x^2 + 3x^3 + ... + (n-1)x^{n-1} + nx^n$

$xS_n = \quad x^2 + 2x^3 + ... + (n-1)x^n + nx^{n+1}$

$S_n - xS_n = x + x^2 + x^3 + ...x^n - nx^{n+1}$

$(1-x)S_n = [x + x^2 + x^3 + ...x^n] - nx^{n+1}$

The expression in the square brackets in a geometric series with $a = x$ and $r = x$.
The sum of these terms,

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ 5

$$\text{Sum}_n = \frac{x(1-x^n)}{1-x}$$

$$\Rightarrow (1-x)S_n = \frac{x(1-x^n)}{1-x} - nx^{n+1} \Rightarrow S_n = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{(1-x)}$$

Editor's note: This is an artithmetic geometric series. The syllabus states that students only need to calculate these to infinity, *not* to the sum of *n* terms.

5 (c) (ii)

$\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$. **Example**: $\lim_{n\to\infty} (\frac{3}{5})^n = 0$

$$S_\infty = \frac{x(1-0)}{(1-x)^2} - \frac{n(0)}{(1-x)} = \frac{x}{(1-x)^2}$$