SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2006

4 (a) $-2+2+6+...+(4n-6)$ are the first n terms of an arithmetic series. S_n, the sum of these n terms, is 160. Find the value of n.

4 (b) The sum to infinity of a geometric series is $\frac{9}{2}$. The second term of the series is -2. Find the vaue of r, the common ratio of the series.

4 (c) The sequence $u_1, u_2, u_3, ...,$ defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3$, is as follows:
3, 9, 21, 45, 93, ...

(i) Find u_6, and verify that it is equal to the sum of the first six terms of a geometric series with first term 3 and common ratio 2.

(ii) Given that, for all k, u_k is the sum of the first k terms of a geometric series with first term 3 and common ratio 2, find $\sum_{k=1}^{n} u_k$.

SOLUTION

4 (a)

Summing formula: $S_n = \frac{n}{2}[2a+(n-1)d]$ 3

$a=-2,\ d=4,\ S_n=160$

$\Rightarrow S_n = \frac{n}{2}[2(-2)+(n-1)(4)] = 160$

$\Rightarrow \frac{n}{2}[4n-8] = 160 \Rightarrow n(2n-4) = 160 \Rightarrow 2n^2 - 4n - 160 = 0$

$\Rightarrow n^2 - 2n - 80 = 0 \Rightarrow (n-10)(n+8) = 0 \Rightarrow n = 10, -8$

Answer: $n = 10$

4 (b)

$S_\infty = \frac{a}{1-r},\ -1 < r < 1$ 6

$S_\infty = \frac{9}{2},\ ar = -2$

$S_\infty = \frac{a}{1-r} = \frac{9}{2} \Rightarrow 2a = 9 - 9r$ ....(**1**)

$ar = -2 \Rightarrow a = -\frac{2}{r}$ ....(**2**)

Substituting equation (**2**) into equation (**1**):

$-\frac{4}{r} = 9 - 9r \Rightarrow -4 = 9r - 9r^2 \Rightarrow 9r^2 - 9r - 4 = 0$

$\Rightarrow (3r-4)(3r+1) = 0 \Rightarrow r = -\frac{1}{3}, \frac{4}{3}$

$r = -\frac{1}{3}$ is the only solution as the sum to infinity formula only applies to values of r between -1 and 1.

4 (c) (i)

$u_{n+1} = 2u_n + 3 \Rightarrow u_6 = 2u_5 + 3 = 2(93) + 5 = 189$

Geometric Series: $a = 3$, $r = 2$, $n = 6$

$$S_6 = \frac{3(1-2^6)}{1-2} = -3(1-64) = -3(-63) = 189$$

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ 5

4 (c) (ii)

$$u_k = \frac{3(1-2^k)}{1-2} = 3(2^k - 1) = 3(2^k) - 3$$

To evaluate $\sum_{k=1}^{n} u_k$ set out a sum table.

SUM TABLE

$k = 1:\ 3(2) - 3$

$k = 2:\ 3(2^2) - 3$

$k = 3:\ 3(2^3) - 3$

⋮

$k = n-1:\ 3(2^{n-1}) - 3$

$k = n:\ 3(2^n) - 3$

$$\therefore \sum_{k=1}^{n} u_k = 3(2 + 2^2 + 2^3 + \ldots + 2^n) - 3n$$

Inside the bracket, you have a geometric series with $a = 2$ and $r = 2$.

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ 5

$$\Rightarrow \sum_{k=1}^{n} u_k = 3\left(\frac{2(1-2^n)}{1-2}\right) - 3n$$

$$\Rightarrow \sum_{k=1}^{n} u_k = 6(2^n - 1) - 3n$$

5 (a) Find the value of the middle term of the binomial expansion of $\left(\dfrac{x}{y} - \dfrac{y}{x}\right)^8$.

5 (b) (i) Express $\dfrac{2}{(r+1)(r+3)}$ in the form $\dfrac{A}{r+1} + \dfrac{B}{r+3}$.

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)}$.

(iii) Hence evaluate $\displaystyle\sum_{r=1}^{\infty} \frac{2}{(r+1)(r+3)}$.

5 (c) (i) Given two real numbers a and b, where $a > 1$ and $b > 1$, prove that

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2.$$

(ii) Under what condition is $\dfrac{1}{\log_b a} + \dfrac{1}{\log_a b} = 2$.

SOLUTION

5 (a)

$n = 8 \Rightarrow$ Middle term: $\binom{8}{4}\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^4 = 70$

NOTE: The middle term is given by $\binom{n}{\frac{n}{2}} x^{\frac{n}{2}} y^{\frac{n}{2}}$.

5 (b) (i)

$$\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3} \Rightarrow \frac{2}{(r+1)(r+3)} = \frac{A(r+3)+B(r+1)}{(r+1)(r+3)}$$

$\Rightarrow 2 = A(r+3) + B(r+1)$

Let $r = -3 \Rightarrow 2 = -2B \Rightarrow B = -1$

Let $r = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$

[This is an identity which means it holds for all values of r. Therefore, you any values of r you wish. Be clever with your choice.]

$$\therefore \frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$$

5 (b) (ii)

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+1} - \frac{1}{n+3}$$

SUM TABLE

$r = 1$: $\frac{1}{2} - \frac{1}{4}$

$r = 2$: $\frac{1}{3} - \frac{1}{5}$

$r = 3$: $\frac{1}{4} - \frac{1}{6}$

⋮

$r = n-1$: $\frac{1}{n} - \frac{1}{n+2}$

$r = n$: $\frac{1}{n+1} - \frac{1}{n+3}$

5 (b) (iii)

$$\sum_{r=1}^{\infty}\left(\frac{5}{6} - \frac{1}{r+2} - \frac{1}{r+1}\right) = \frac{5}{6}$$

5 (c) (i)

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2 \Rightarrow \log_a b + \frac{1}{\log_a b} \geq 2$$ [Multiply across by $\log_a b$]

$$\Rightarrow (\log_a b)^2 + 1 \geq 2\log_a b \Rightarrow (\log_a b)^2 - 2\log_a b + 1 \geq 0$$

$$\Rightarrow (\log_a b - 1)(\log_a b - 1) \geq 0 \Rightarrow (\log_a b - 1)^2 \geq 0$$ [This is always true]

5 (c) (ii)

$$\frac{1}{\log_b a} + \frac{1}{\log_a b} = 2 \Rightarrow \log_a b + \frac{1}{\log_a b} = 2$$

Let $u = \log_a b \Rightarrow u + \frac{1}{u} = 2 \Rightarrow u^2 - 2u + 1 = 0$

$\Rightarrow (u-1)(u-1) = 0 \Rightarrow u = 1$

$\Rightarrow \log_a b = 1 \Rightarrow b = a$