## 2005

4 (a) Write the recurring decimal 0·636363.... as an infinite geometric series and hence as a fraction.

4 (b) (i) The first three terms in the binomial expansion of $(1+kx)^n$ are $1-21x+189x^2$. Find the value of $n$ and the value of $k$.

(ii) A sequence is defined by $u_n=(2-n)2^{n-1}$. Show that $u_{n+2}-4u_{n+1}+4u_n=0$, for all $n\in\mathbf{N}$.

4 (c) (i) Show that $\frac{a+b}{2}\le\sqrt{\frac{a^2+b^2}{2}}$, where $a$ and $b$ are real numbers.

(ii) The lengths of the sides of a right-angled triangle are $a$, $b$ and $c$, where $c$ is the length of the hypotenuse. Using the result from part (**i**), or otherwise, show that $a+b\le c\sqrt{2}$.

### SOLUTION

**4 (a)**

$0\cdot636363...=\frac{63}{100}+\frac{63}{10000}+\frac{63}{1000000}+...=63(\frac{1}{100}+\frac{1}{10000}+\frac{1}{1000000}+...)$

Infinite geometric series: $a=\frac{1}{100}$, $r=\frac{1}{100}$

$$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{100}}{1-\frac{1}{100}}=\frac{\frac{1}{100}}{\frac{99}{100}}=\frac{1}{99}$$

$\therefore 0\cdot636363....=63(\frac{1}{99})=\frac{63}{99}$

**4 (b) (i)**

$$(1+kx)^n=\binom{n}{0}(1)^n(kx)^0+\binom{n}{1}(1)^{n-1}(kx)^1+\binom{n}{2}(1)^{n-2}(kx)^2+....=1-21x+189x^2+...$$

$$\Rightarrow 1+nkx+\frac{n(n-1)}{2}k^2x^2+....=1-21x+189x^2+...$$

Lining up coefficients: $nk=-21\Rightarrow k=-\frac{21}{n}$ and $\frac{n(n-1)}{2}k^2=189$

$\Rightarrow 441n-441=378n\Rightarrow 63n=441\Rightarrow n=7\Rightarrow k=-3$

**4 (b) (ii)**

$u_n = (2-n)2^{n-1} \Rightarrow u_{n+2} = (-n)2^{n+1}$

$u_n = (2-n)2^{n-1} \Rightarrow u_{n+1} = (1-n)2^{n}$

$\therefore u_{n+2} - 4u_{n+1} + 4u_n = (-n)2^{n+1} - 4(1-n)2^n + 4(2-n)2^{n-1}$

$= 2^{n-1}[(-n)2^2 - 4(1-n)2^1 + 4(2-n)] = 2^{n-1}[-4n - 8 + 8n + 8 - 4n] = 2^{n-1}[0] = 0$

**4 (c) (i)**

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \Rightarrow \left(\frac{a+b}{2}\right)^2 \le \frac{a^2+b^2}{2}$$

$$\Rightarrow \frac{a^2+2ab+b^2}{4} \le \frac{a^2+b^2}{2} \Rightarrow a^2+2ab+b^2 \le 2a^2+2b^2$$

$\Rightarrow 0 \le a^2 - 2ab + b^2 \Rightarrow (a-b)^2 \ge 0$

**4 (c) (ii)**

Pythagoras: $a^2 + b^2 = c^2$

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \Rightarrow \frac{a+b}{2} \le \sqrt{\frac{c^2}{2}} \Rightarrow a+b \le c\sqrt{2}$$

5 (a) Solve for $x$: $\sqrt{10-x} = 4-x$.

5 (b) Prove by induction that $\displaystyle\sum_{r=1}^{n}(3r-2) = \frac{n}{2}(3n-1)$.

5 (c) (i) Show that $\dfrac{1}{\log_a b} = \log_b a$, where $a, b > 0$ and $a, b \ne 1$.

(ii) Show that $\dfrac{1}{\log_2 c} + \dfrac{1}{\log_3 c} + \dfrac{1}{\log_4 c} + \ldots\ldots + \dfrac{1}{\log_r c} = \dfrac{1}{\log_{r!} c}$, where $c > 0$, $c \ne 1$.

**SOLUTION**

**5 (a)**

$\sqrt{10-x} = 4-x \Rightarrow 10-x = (4-x)^2$ [Squaring both sides.]

$\Rightarrow 10 - x = 16 - 8x + x^2 \Rightarrow x^2 - 7x + 6 = 0$

$\Rightarrow (x-6)(x-1) = 0 \Rightarrow x = 6, 1$

Check both solutions:

$x = 6$: $\sqrt{10-6} = 4-6 \Rightarrow \sqrt{4} = -2$ [Wrong]

$x = 1$: $\sqrt{10-1} = 4-1 \Rightarrow \sqrt{9} = 3$ [Correct]

**5 (b)**

**STEPS**
**1**. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
**2**. Assuming result is true for $n = k$.
**3**. Prove result is true for $n = (k+1)$.

**SUMMATIONS:** When proving a summation always expand as follows:

$$\sum_{r=1}^{k} f(r) = \{f(1) + f(2) + ...... + f(k)\}$$

$$\sum_{r=1}^{k+1} f(r) = \{f(1) + f(2) + ...... + f(k)\} + f(k+1)$$

**1.** Prove it is true for $n = 1$: $\sum_{r=1}^{1}(3r-2) = 3(1) - 2 = 1$ and $\frac{n}{2}(3n-1) = \frac{1}{2}(3(1)-1) = 1$

**2.** Assume it is true for $n = k$: $\sum_{r=1}^{k}(3r-2) = \{1+4+7+......+3k-2\} = \frac{k}{2}(3k-1)$

**3.** Prove it is true for $n = k + 1$:

You need to prove that $\sum_{r=1}^{k+1}(3r-2) = \{1+4+7+......+3k-2\} + (3k+1) = \frac{k+1}{2}(3k+2)$

Using the result in step 2 $\Rightarrow \frac{k}{2}(3k-1) + (3k+1) = \frac{k+1}{2}(3k+2)$

$\Rightarrow k(3k-1) + 2(3k+1) = (k+1)(3k+2)$

$\Rightarrow 3k^2 - k + 6k + 2 = (k+1)(3k+2) \Rightarrow 3k^2 + 5k + 2 = (k+1)(3k+2)$

$\Rightarrow (k+1)(3k+2) = (k+1)(3k+2)$

**5 (c) (i)**

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$$

**LOG RULES**

$\log_a M = \frac{\log_b M}{\log_b a}$ [Used to change base]

$\log_a a = 1$

**5 (c) (ii)**

$$\frac{1}{\log_2 c} + \frac{1}{\log_3 c} + \frac{1}{\log_4 c} + ....... + \frac{1}{\log_r c} = \log_c 2 + \log_c 3 + \log_c 4 + ....... + \log_c r$$

$$= \log_c(2 \times 3 \times 4 \times .... \times r) = \log_c(r!) = \frac{1}{\log_{r!} c}$$