SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2004

4 (a) Show that $3\binom{n}{3}=n\binom{n-1}{2}$ for all natural numbers $n \geq 3$.

4 (b) (i) Show that $\frac{2}{(2r-1)(2r+1)}=\frac{1}{2r-1}-\frac{1}{2r+1}$, $r \neq \pm\frac{1}{2}$.

(ii) Hence, find $\sum_{r=1}^{n}\frac{2}{(2r-1)(2r+1)}$.

(iii) Evaluate $\sum_{r=1}^{\infty}\frac{2}{(2r-1)(2r+1)}$.

4 (c) (i) The sequence $u_1, u_2, u_3, ...$ is given by $u_{n+1}=\sqrt{4-(u_n)^2}$ and $u_1 = a > 0$. For what value of a will all the terms of the sequence be equal to each other?

(ii) p, q and r are three numbers in arithmetic sequence. Prove that $p^2+r^2 \geq 2q^2$.

SOLUTION

4 (a)

To prove: $3\binom{n}{3}=n\binom{n-1}{2}$

LHS

$$3\binom{n}{3}=3\times\frac{n(n-1)(n-2)}{3\times2\times1}=\frac{n(n-1)(n-2)}{2}$$

RHS

$$n\binom{n-1}{2}=n\times\frac{(n-1)(n-2)}{2}$$

4 (b) (i)

$$\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{1(2r+1)-1(2r-1)}{(2r-1)(2r+1)}=\frac{2r+1-2r+1}{(2r-1)(2r+1)}=\frac{2}{(2r-1)(2r+1)}$$

4 (b) (ii)

$$\sum_{r=1}^{n}\frac{2}{(2r-1)(2r+1)}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$

$$=1-\frac{1}{2n-1}$$

SUM TABLE

$r=1$: $\frac{1}{1}-\frac{1}{3}$

$r=2$: $\frac{1}{3}-\frac{1}{5}$

$r=n-1$: $\frac{1}{2n-3}-\frac{1}{2n-1}$

$r=n$: $\frac{1}{2n-1}-\frac{1}{2n+1}$

4 (b) (iii)

$$\sum_{r=1}^{\infty}\frac{2}{(2r-1)(2r+1)}=1$$

4 (c) (i)

$u_{n+1} = \sqrt{4-(u_n)^2} \Rightarrow u_2 = \sqrt{4-u_1^2}$

$u_1 = a \Rightarrow a = \sqrt{4-a^2}$ [As all the terms are equal $u_1 = u_2 = a$]

$\therefore a^2 = 4-a^2 \Rightarrow 2a^2 = 4 \Rightarrow a^2 = 2 \Rightarrow a = \sqrt{2}$ [Remember $a > 0$]

4 (c) (ii)

$p, q, r \to a, a+d, a+2d$ [Arithmetic sequence]

$p^2 + r^2 \geq 2q^2 \Rightarrow a^2 + (a+2d)^2 \geq 2(a+d)^2$

$\Rightarrow a^2 + a^2 + 4ad + 4d^2 \geq 2(a^2 + 2ad + d^2)$

$\Rightarrow a^2 + a^2 + 4ad + 4d^2 \geq 2a^2 + 4ad + 2d^2$

$\Rightarrow 2d^2 \geq 0$ [This is always true.]

5 (a) Find the fifth term in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$ and show that it is independent of x.

5 (b) (i) In a geometric series, the second term is 8 and the fifth term is 27. Find the first term and the common ratio.

(ii) Solve $\log_4 x - \log_4(x-2) = \frac{1}{2}$.

5 (c) Prove by induction that $2^n \geq n^2$, $n \in \mathbf{N}$, $n \geq 4$.

SOLUTION

5 (a)

$u_5 = ?,\ r = 4,\ n = 6$

$u_{r+1} = {}^nC_r (x)^{n-r} (y)^r = \binom{n}{r} (x)^{n-r} (y)^r$ **10**

$u_5 = \binom{6}{4} (x^2)^2 \left(-\frac{1}{x}\right)^4 = 15$

5 (b) (i)

$\dfrac{ar^4 = 27}{ar = 8} \Rightarrow r^3 = \frac{27}{8} \Rightarrow r = \frac{3}{2}$

$ar = 8 \Rightarrow a(\frac{3}{2}) = 8 \Rightarrow a = \frac{16}{3}$

The forty-third term of a geometric sequence is written as $u_{43} = ar^{42}$

5 (b) (ii)

LOG RULES

2. $\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)$

$\log_4 x - \log_4(x-2) = \frac{1}{2} \Rightarrow \log_4 \left(\frac{x}{x-2}\right) = \frac{1}{2}$

$\Rightarrow \left(\frac{x}{x-2}\right) = 4^{\frac{1}{2}} = 2 \Rightarrow x = 2(x-2) \Rightarrow x = 2x - 4 \Rightarrow x = 4$

3 (c)

Steps

1. Prove result is true for some starting value of $n \in \mathbf{N}_0$.

2. Assuming result is true for $n = k$.

3. Prove result is true for $n = (k+1)$.

Prove $2^n \geq n^2$ for all $n \geq 4$.

Rewrite it as: Prove $n^2 \leq 2^n$ for all $n \geq 4$.

1. Prove this statement is true for $n = 4$.

2. Assume it is true for $n = k \Rightarrow k^2 \leq 2^k$

3. Prove for $n = k + 1$. Show that $\Rightarrow (k+1)^2 \leq 2^{k+1} \Rightarrow k^2 (1+\frac{1}{k})^2 \leq 2^k \times 2$

From Step 2: $k^2 \leq 2^k$

$k \geq 4 \Rightarrow \frac{1}{k} \leq \frac{1}{4} \Rightarrow 1+\frac{1}{k} \leq \frac{5}{4}$

$\Rightarrow (1+\frac{1}{k})^2 \leq \frac{25}{16} \leq 2$