SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2003

4 (a) Express the recurring decimal 0·252525... in the form $\frac{p}{q}$ where $p, q \in \mathbf{N}$ and $q \neq 0$.

4 (b) In an arithmetic series, the sum of the second term and the fifth term is 18. The sixth term is greater than the third term by 9.

(i) Find the first term and the common difference.

(ii) What is the smallest value of n such that $S_n > 600$, where S_n is the sum of the first n terms of the series?

4 (c) (i) $u_1, u_2, u_3, u_4, u_5, \ldots\ldots$ is a sequence where $u_1 = 2$ and $u_{n+1} = (-1)^n u_n + 3$. Evaluate u_2, u_3, u_4, u_5 and u_{10}.

(ii) a, b, c, d are the first, second, third and fourth terms of a geometric sequence, respectively. Prove that $a^2 - b^2 - c^2 + d^2 \geq 0$.

SOLUTION

4 (a)

$$0\cdot252525\ldots = \tfrac{25}{100} + \tfrac{25}{10000} + \tfrac{25}{1000000} + \ldots = 25(\tfrac{1}{100} + \tfrac{1}{10000} + \tfrac{1}{1000000} + \ldots)$$

Infinite geometric series: $a = \frac{1}{100}$, $r = \frac{1}{100}$

$S_\infty = \dfrac{a}{1-r}, \ -1 < r < 1$ 6

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99}{100}} = \frac{1}{99}$$

$$\therefore 0\cdot252525\ldots = 25(\tfrac{1}{99}) = \tfrac{25}{99}$$

4 (b) (i)

General term: $u_n = a + (n-1)d$ 2

Summing formula: $S_n = \frac{n}{2}[2a + (n-1)d]$ 3

The fifty-sixth term of an arithmetic sequence: $u_{56} = a + 55d$

$u_2 = a + d$

$u_3 = a + 2d$

$u_5 = a + 4d$

$u_6 = a + 5d$

$u_2 + u_5 = 18 \Rightarrow a + d + a + 4d = 18 \Rightarrow 2a + 5d = 18 \ldots.(\mathbf{1})$

$u_6 = u_3 + 9 \Rightarrow a + 5d = a + 2d + 9 \Rightarrow 3d = 9 \Rightarrow d = 3 \ldots.(\mathbf{2})$

Substituting the value for d into equation (**2**): $\Rightarrow 2a + 5(3) = 18 \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

4 (b) (ii)

$$S_n = \tfrac{n}{2}[2a + (n-1)d] = 600 \Rightarrow \tfrac{n}{2}[2(\tfrac{3}{2}) + (n-1)(3)] = 600$$

$$\Rightarrow \tfrac{n}{2}[3 + 3n - 3] = 600 \Rightarrow 3n^2 = 1200 \Rightarrow n^2 = 400 \Rightarrow n = 20$$

The question asks what is the smallest value of n for the sum to exceed 600. 21 terms are needed to exceed this value.

ANSWER: $n = 21$

4 (c) (i)

$u_1 = 2$

$u_2 = (-1)^1 u_1 + 3 = -2 + 3 = 1$

$u_3 = (-1)^2 u_2 + 3 = 1 + 3 = 4$

$u_4 = (-1)^3 u_3 + 3 = -4 + 3 = -1$

$u_5 = (-1)^4 u_4 + 3 = -1 + 3 = 2$ [The sequence starts repeating.]

$2, 1, 4, -1, 2, 1, 4, -1, 2, 1, 4, -1, ...$

As can be seen the tenth term $u_{10} = 1$.

4 (c) (ii)

$a, b, c, d \rightarrow a, ar, ar^2, ar^3$ [Terms of a geometric sequence]

$a^2 - b^2 - c^2 + d^2 \geq 0 \Rightarrow a^2 - a^2r^2 - a^2r^4 + a^2r^6 \geq 0$

$\Rightarrow a^2(1 - r^2 - r^4 + r^6) \geq 0 \Rightarrow 1 - r^2 - r^4 + r^6 \geq 0$

$\Rightarrow 1(1 - r^2) - r^4(1 - r^2) \geq 0 \Rightarrow (1 - r^4)(1 - r^2) \geq 0$

$\Rightarrow (1 - r^2)(1 + r^2)(1 - r^2) \geq 0 \Rightarrow (1 - r^2)^2(1 + r^2) \geq 0$ [This is true for all values of *r*.]

5 (a) Solve for *x*: $x = \sqrt{7x - 6} + 2$.

5 (b) Use induction to prove that 8 is a factor of $7^{2n+1} + 1$ for any positive integer *n*.

5 (c) Consider the binomial expansion of $\left(ax + \frac{1}{bx}\right)^8$, where *a* and *b* are non-zero real numbers.

(i) Write down the general term.

(ii) Given that the coefficient of x^2 is the equal to the coefficient of x^4, show that $ab = 2$.

Solution

5 (a)

$x = \sqrt{7x - 6} + 2 \Rightarrow (x - 2) = \sqrt{7x - 6}$ [Isolate the surd expression.]

$\Rightarrow (x - 2)^2 = 7x - 6 \Rightarrow x^2 - 4x + 4 = 7x - 6$

$\Rightarrow x^2 - 11x + 10 = 0 \Rightarrow (x - 10)(x - 1) = 0 \Rightarrow x = 1, 10$

Check solutions:

$x = 1$: $1 = \sqrt{7(1) - 6} + 2 \Rightarrow 1 = \sqrt{1} + 2 \Rightarrow 1 = 1 + 2$ [Not a solution]

$x = 10$: $10 = \sqrt{7(10) - 6} + 2 \Rightarrow 10 = \sqrt{64} + 2 \Rightarrow 10 = 8 + 2$ [Works]

Answer: $x = 10$

5 (b)

> **STEPS**
> **1**. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
> **2**. Assuming result is true for $n = k$.
> **3**. Prove result is true for $n = (k+1)$.

1. Prove for $n = 1$.

$7^{2(1)+1} + 1 = 7^3 + 1 = 343 + 1 = 344$

$344 \div 8 = 43$ [Therefore, true for $n = 1$.]

2. Assume for $n = k \Rightarrow 7^{2k+1} + 1 = 8m,\ m \in \mathbf{N}_0$.

3. Prove for $n = k + 1$.

$\Rightarrow 7^{2(k+1)+1} + 1 = 7^{2k+3} + 1 = 7^2(7^{2k+1}) + 1 = 49(7^{2k+1}) + 1$

From step 2: $7^{2k+1} = 8m - 1$

$\Rightarrow 7^{2k+3} + 1 = 49(8m-1) + 1 = 49(8m) + 48 = 8(49m + 6) = 8a,\ a \in \mathbf{N}_0$.

5 (c) (i)

General term of $\left(ax + \frac{1}{bx}\right)^8$

$u_{r+1} = {}^nC_r\,(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r$ **10**

$n = 8$, $r = ?$

$$u_{r+1} = \binom{8}{r}(ax)^{8-r}\left(\frac{1}{bx}\right)^r$$

5 (c) (ii)

Tidy up the answer above $\Rightarrow u_{r+1} = \binom{8}{r}\frac{a^{8-r}x^{8-r}}{b^r x^r} = \binom{8}{r}\left(\frac{a^{8-r}}{b^r}\right)x^{8-2r}$

x^2 term: $8 - 2r = 2 \Rightarrow r = 3$

x^4 term: $8 - 2r = 4 \Rightarrow r = 2$

Coefficient of x^2 = Coefficient of x^4 $\Rightarrow \binom{8}{3}\left(\frac{a^5}{b^3}\right) = \binom{8}{2}\left(\frac{a^6}{b^2}\right)$

$$\Rightarrow 56\left(\frac{a^5}{b^3}\right) = 28\left(\frac{a^6}{b^2}\right) \Rightarrow 2\left(\frac{1}{b}\right) = a \Rightarrow ab = 2$$