# SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

## 2002

4 (a) Find in terms of $n$, the sum of the first $n$ terms of the geometric series $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\ldots$

4 (b) (i) Show that $\dfrac{2}{k(k+2)}=\dfrac{1}{k}-\dfrac{1}{k+2}$, for all $k\in\mathbf{R}$, $k\neq 0,\,-2$.

(ii) Evaluate, in terms of $n$, $\displaystyle\sum_{k=1}^{n}\frac{2}{k(k+2)}$.

(iii) Evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k(k+2)}$.

4 (c) Three numbers are in arithmetic sequence. Their sum is 27 and their product is 704.
Find the three numbers.

**SOLUTION**

**4 (a)**

Summing formula: $S_n=\dfrac{a(1-r^n)}{(1-r)}$ ....... 5

$a=3,\ r=\frac{1}{2}$

$$\therefore S_n=\frac{3(1-(\frac{1}{2})^n)}{1-\frac{1}{2}}=6(1-(\tfrac{1}{2})^n)$$

**4 (b) (i)**

$$\frac{1}{k}-\frac{1}{k+2}=\frac{1(k+2)-k}{k(k+2)}=\frac{2}{k(k+2)}$$

**4 (b) (ii)**

$$\sum_{k=1}^{n}\frac{2}{k(k+2)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)$$

$$=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$=\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

**SUM TABLE**

$k=1$: $\dfrac{1}{1}-\cancel{\dfrac{1}{3}}$

$k=2$: $\dfrac{1}{2}-\cancel{\dfrac{1}{4}}$

$k=3$: $\cancel{\dfrac{1}{3}}-\cancel{\dfrac{1}{5}}$

$k=n-1$: $\cancel{\dfrac{1}{n-1}}-\dfrac{1}{n+1}$

$k=n$: $\cancel{\dfrac{1}{n}}-\dfrac{1}{n+2}$

**4 (b) (iii)**

$$\sum_{k=1}^{\infty}\frac{2}{k(k+2)}=\frac{3}{2}$$

**4 (c)**

Call the numbers $a-d,\ a,\ a+d$

> If you are asked to choose three consecutive terms in an arithmetic sequence choose them as: $a-d,\ a,\ a+d$

Sum: $3a = 27 \Rightarrow a = 9$

Product: $(a-d)a(a+d) = 704 \Rightarrow (9-d)9(9+d) = 704$

$\Rightarrow 81 - d^2 = \frac{704}{9} \Rightarrow d^2 = 81 - \frac{704}{9} = \frac{25}{9} \Rightarrow d = \pm\frac{5}{3}$

Therefore, the 3 numbers are: $a-d,\ a,\ a+d = 9-\frac{5}{3},\ 9,\ 9+\frac{5}{3} = \frac{22}{3},\ 9,\ \frac{32}{3}$

**Note**: There are two values of $d$. Choosing either value gives you the same three numbers in a different order.

5 (a) Find the value of $x$ in each case:

(i) $\dfrac{8}{2^x} = 32$

(ii) $\log_9 x = \dfrac{3}{2}$.

5 (b) The first three terms in the binomial expansion of $(1+ax)^n$ are $1 + 2x + \frac{7}{4}x^4$.

(i) Find the value of $a$ and the value of $n$.

(ii) Hence, find the middle term in the expansion.

5 (c) Prove by induction that, for any positive integer n, $x + x^2 + x^3 + ... + x^n = \dfrac{x(x^n - 1)}{x-1}$,

where $x \neq 1$.

**Solution**

**5 (a) (i)**

$$\frac{8}{2^x} = 32 \Rightarrow \frac{2^3}{2^x} = 2^5$$

$$\Rightarrow 2^{3-x} = 2^5 \Rightarrow 3 - x = 5 \Rightarrow x = -2$$

**5 (a) (ii)**

$$\log_9 x = \frac{3}{2} \Rightarrow x = 9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$$

**5 (b) (i)**

You need to remember the formula for expanding binomials, especially the first three terms.

$$(x+y)^n = \binom{n}{0}(x)^n(y)^0 + \binom{n}{1}(x)^{n-1}(y)^1 + \binom{n}{2}(x)^{n-2}(y)^2 ....... \quad ....... \ \mathbf{9}$$

$$(1+ax)^n = 1 + 2x + \tfrac{7}{4}x^2 + .... \Rightarrow \binom{n}{0}(1)^n(ax)^0 + \binom{n}{1}(1)^{n-1}(ax)^1 + \binom{n}{2}(1)^{n-2}(ax)^2 = 1 + 2x + \tfrac{7}{4}x^2$$

$$\Rightarrow 1 + nax + \frac{n(n-1)}{2}a^2x^2 = 1 + 2x + \tfrac{7}{4}x^2$$

Lining up coefficients: $na = 2 \Rightarrow a = \frac{2}{n}$

$$\frac{n(n-1)}{2}a^2 = \frac{7}{4} \Rightarrow \frac{n(n-1)}{2}\left(\frac{2}{n}\right)^2 = \frac{7}{4} \Rightarrow \frac{n(n-1)}{2} \times \frac{4}{n^2} = \frac{7}{4}$$

$$\Rightarrow (n-1) \times \frac{2}{n} = \frac{7}{4} \Rightarrow 8(n-1) = 7n \Rightarrow 8n - 8 = 7n$$

$$\Rightarrow n = 8 \Rightarrow a = \tfrac{2}{n} = \tfrac{2}{8} = \tfrac{1}{4}$$

**5 (b) (ii)**

Middle term of $(1 + \frac{1}{4}x)^8$ is

$$\binom{8}{4}(1)^4(\tfrac{1}{4}x)^4 = \frac{70x^4}{256} = \frac{35x^4}{128}$$

The middle term is given by $\binom{n}{\frac{n}{2}}x^{\frac{n}{2}}y^{\frac{n}{2}}$.

**5 (c)**

**Steps**

**1**. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
**2**. Assuming result is true for $n = k$.
**3**. Prove result is true for $n = (k+1)$.

Rewrite as: $\displaystyle\sum_{r=1}^{n} x^r = \frac{x(x^n - 1)}{x-1}$

**1**. Prove for $n = 1$: $\displaystyle\sum_{r=1}^{1} x^r = x^1 = x$ and $\displaystyle\frac{x(x^n-1)}{x-1} = \frac{x(x-1)}{x-1} = x$ [True for $n = 1$]

**2**. Assume for $n = k$: $\displaystyle\sum_{r=1}^{k} x^r = \{x + x^2 + x^3 + ... + x^k\} = \frac{x(x^k - 1)}{x-1}$

**3**. Prove for $n = k+1$: $\displaystyle\sum_{r=1}^{k+1} x^r = \{x + x^2 + x^3 + ... + x^k\} + x^{k+1} = \frac{x(x^{k+1}-1)}{x-1}$

Using step 2: $\displaystyle\Rightarrow \frac{x(x^k-1)}{x-1} + x^{k+1} = \frac{x(x^{k+1}-1)}{x-1} \Rightarrow \frac{x(x^k-1)x^{k+1}(x-1)}{x-1} = \frac{x(x^{k+1}-1)}{x-1}$

$$\Rightarrow \frac{x(x^k-1) + x^{k+1}(x-1)}{x-1} = \frac{x(x^{k+1}-1)}{x-1}$$

$$\Rightarrow \frac{x^{k+1} - x + x^{k+2} - x^{k+1}}{x-1} = \frac{x(x^{k+1}-1)}{x-1} \Rightarrow \frac{x^{k+2} - x}{x-1} = \frac{x(x^{k+1}-1)}{x-1} \Rightarrow \frac{x(x^{k+1}-1)}{x-1} = \frac{x(x^{k+1}-1)}{x-1}$$

Therefore, true for $n = k + 1$.