SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2001

4 (a) The sum of the first n terms of an arithmetic series is given by $S_n = 3n^2 - 4n$. Use S_n to find: (i) the first term, u_1

(ii) the sum of the second term and the third term, $u_2 + u_3$.

4 (b) (i) Show that $\dfrac{1}{(n+2)(n+2)} = \dfrac{1}{n+2} - \dfrac{1}{n+3}$ for $n \in \mathbf{N}$.

(ii) Hence, find $\displaystyle\sum_{n=1}^{k} \frac{1}{(n+2)(n+2)}$ and evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+2)}$.

4 (c) (i) Write $\dfrac{n^3+8}{n+2}$ in the form $an^2 + bn + c$ where $a, b, c \in \mathbf{R}$.

(ii) Hence, evaluate $\displaystyle\sum_{n=1}^{30} \frac{n^3+8}{n+2}$.

[NOTE: $\displaystyle\sum_{n=1}^{k} n = \frac{k}{2}(k+1); \sum_{n=1}^{k} n^2 = \frac{k}{6}(k+1)(2k+1).$]

SOLUTION

4 (a) (i)

$$S_n = 3n^2 - 4n \Rightarrow S_1 = u_1 = 3(1)^2 - 4(1) = 3 - 4 = -1$$

4 (a) (ii)

$$u_2 + u_3 = S_3 - S_1 = 3(3)^2 - 4(3) - (-1)$$
$$= 27 - 12 + 1 = 16$$

$u_n = S_n - S_{n-1}$ 1

4 (b) (i)

$$\frac{1}{n+2} - \frac{1}{n+3} = \frac{1(n+3) - 1(n+2)}{(n+2)(n+3)} = \frac{n+3-n-2}{(n+2)(n+3)} = \frac{1}{(n+2)(n+3)}$$

4 (b) (ii)

$$\sum_{n=1}^{k} \frac{1}{(n+2)(n+3)} = \sum_{n=1}^{k}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$= \frac{1}{3} - \frac{1}{k+3}$$

SUM TABLE

$n = 1$:	$\frac{1}{3} - \frac{1}{4}$
$n = 2$:	$\frac{1}{4} - \frac{1}{5}$
$n = k-1$:	$\frac{1}{k+1} - \frac{1}{k+2}$
$n = k$:	$\frac{1}{k+2} - \frac{1}{k+3}$

4 (b) (ii)

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3}$$

4 (c) (i)

$$\frac{n^3+8}{n+2}=\frac{(n)^3+(2)^3}{(n+2)}=\frac{(n+2)(n^2-2n+4)}{(n+2)}=n^2-2n+4$$

Sum of 2 cubes

$a^3+b^3=(a+b)(a^2-ab+b^2)$ 2

4 (c) (ii)

$\sum_{r=1}^{n} r = S_n = 1+2+....+n=\frac{n}{2}(n+1)$ 7

$\sum_{r=1}^{n} r^2 = S_n = 1^2+2^2+......+n^2=\frac{n}{6}(n+1)(2n+1)$ 8

$$\sum_{n=1}^{30}\frac{n^3+8}{n+2}=\sum_{n=1}^{30}(n^2-2n+4)=\sum_{n=1}^{30}n^2-2\sum_{n=1}^{30}n+4\sum_{n=1}^{30}1$$

$=\frac{30}{6}(30+1)(2(30)+1)-2\times\frac{30}{2}(30+1)+4\times 30$

$=5(31)(61)-30(31)+120=8,645$

5 (a) The second term, u_2, of a geometric sequence is 21. The third term, u_3, is –63. Find

(i) the common ratio

(ii) the first term.

5 (b) (i) Solve $\log_6(x+5)=2-\log_6 x$ for $x > 0$.

(ii) In the binomial expansion of $(1+kx)^6$, the coefficient of x^4 is 240. Find the two possible values of k.

5 (c) Use induction to prove that, for n a positive integer, $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ for all $\theta\in\mathbf{R}$ and $i^2=-1$.

SOLUTION

5 (a) (i)

$$\frac{u_3=ar^2=-63}{u_2=ar=21}\quad \text{Dividing} \Rightarrow r=-3$$

The forty-third term of a geometric sequence is written as $u_{43}=ar^{42}$

5 (a) (ii)

$ar=21\Rightarrow a=\frac{21}{r}=\frac{21}{-3}=-7$

5 (b) (i)

$\log_6(x+5)=2-\log_6 x\Rightarrow \log_6(x+5)+\log_6 x=2$

$\Rightarrow \log_6 x(x+5)=2\Rightarrow x(x+5)=6^2=36$

$\Rightarrow x^2+5x-36=0\Rightarrow(x-4)(x+9)=0\Rightarrow x=4,\,-9$

Check both solution. Only x = 4 works. $x=-9$ gives negative logs which are not allowed.

5 (b) (ii)

Write the general term of $(1+kx)^6$.

$u_{r+1} = {}^nC_r(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r$ **10**

$$u_{r+1} = \binom{6}{r}(1)^{6-r}(kx)^r = \binom{6}{r}k^r x^r$$

As can be seen, $r = 4$ in the term with x^4.

$$\Rightarrow \binom{6}{4}k^4 = 240 \Rightarrow 15k^4 = 240 \Rightarrow k^4 = 16 \Rightarrow k = \pm 2$$

5 (c)

STATEMENT OF DE MOIVRE'S THEOREM

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbf{N}_0$.

PROOF

1. For $n = 1$: Prove $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

i.e. $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$. This is obviously true.

2. For $n = k$: Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

3. For $n = k + 1$: Prove $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

PROOF: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ using **STEP 2**

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.