## 2011

**4.** **(a)** In an arithmetic sequence, the third term is –3 and the sixth term is –15. Find the first term and the common difference.

**(b)** Let $u_n = l(\frac{1}{2})^n + m(-1)^n$ for all $n \in \mathbb{N}$.

**(i)** Verify that $u_n$ satisfies the equation $2u_{n+2} + u_{n+1} - u_n = 0$.

**(ii)** If $a_k = u_k + u_{k+1}$, express $a_k$ in terms of $k$ and $l$.

**(iii)** Find $\sum_{k=1}^{\infty} a_k$, in terms of $l$.

**(iv)** For $l > 0$, find the least positive integer $n$ for which

$$\sum_{k=1}^{n} a_k > (0.99)\sum_{k=1}^{\infty} a_k.$$

**SOLUTION**

**4 (a)**

General term: $\boxed{u_n = a + (n-1)d}$

The fifty-sixth term of an arithmetic sequence: $u_{56} = a + 55d$

$u_3 = -3 \Rightarrow a + 2d = -3........(\mathbf{1})$
$u_6 = -15 \Rightarrow a + 5d = -15....(\mathbf{2})$

$a + 2d = -3......(\mathbf{1})(\times -1)$
$a + 5d = -15....(\mathbf{2})$

$$\begin{array}{r} -a - 2d = 3 \\ a + 5d = -15 \\ \hline 3d = -12 \Rightarrow d = -4 \end{array}$$

Substitute this value of $d$ into Eqn (**1**):

$a + 2(-4) = -3$
$a - 8 = -3$
$\therefore a = 5$

**4 (b) (i)**

$$u_n = l(\tfrac{1}{2})^n + m(-1)^n$$
$$u_{n+1} = l(\tfrac{1}{2})^{n+1} + m(-1)^{n+1}$$
$$u_{n+2} = l(\tfrac{1}{2})^{n+2} + m(-1)^{n+2}$$

$$\begin{aligned}
&2u_{n+2} + u_{n+1} - u_n \\
&= 2[l(\tfrac{1}{2})^{n+2} + m(-1)^{n+2}] + [l(\tfrac{1}{2})^{n+1} + m(-1)^{n+1}] - [l(\tfrac{1}{2})^n + m(-1)^n] \\
&= 2l(\tfrac{1}{2})^{n+2} + 2m(-1)^{n+2} + l(\tfrac{1}{2})^{n+1} + m(-1)^{n+1} - l(\tfrac{1}{2})^n - m(-1)^n \\
&= 2l(\tfrac{1}{2})^{n+2} + l(\tfrac{1}{2})^{n+1} - l(\tfrac{1}{2})^n + 2m(-1)^{n+2} + m(-1)^{n+1} - m(-1)^n \\
&= (\tfrac{1}{2})^n[2l(\tfrac{1}{2})^2 + l(\tfrac{1}{2})^1 - l] + (-1)^n[2m(-1)^2 + m(-1)^1 - m] \\
&= (\tfrac{1}{2})^n[2l(\tfrac{1}{4}) + l(\tfrac{1}{2}) - l] + (-1)^n[2m - m - m] \\
&= (\tfrac{1}{2})^n[\tfrac{1}{2}l + \tfrac{1}{2}l - l] + (-1)^n[2m - m - m] \\
&= (\tfrac{1}{2})^n[0] + (-1)^n[0] \\
&= 0
\end{aligned}$$

**4 (b) (ii)**

$$\begin{aligned}
a_k &= u_k + u_{k+1} \\
&= l(\tfrac{1}{2})^k + m(-1)^k + l(\tfrac{1}{2})^{k+1} + m(-1)^{k+1} \\
&= l(\tfrac{1}{2})^k + l(\tfrac{1}{2})^{k+1} + m(-1)^k + m(-1)^{k+1} \\
&= (\tfrac{1}{2})^k[l + l(\tfrac{1}{2})^1] + (-1)^k[m + m(-1)^1] \\
&= (\tfrac{1}{2})^k[l + \tfrac{1}{2}l] + (-1)^k[m - m] \\
&= (\tfrac{1}{2})^k[\tfrac{3}{2}l] + (-1)^k[0] \\
&= \tfrac{3}{2}l(\tfrac{1}{2})^k
\end{aligned}$$

**4 (b) (iii)**

$$\sum_{k=1}^{\infty} a_k = \tfrac{3}{2}l\sum_{k=1}^{\infty}(\tfrac{1}{2})^k$$
$$= \tfrac{3}{2}l[(\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots\ldots\ldots] \longrightarrow (\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots$$
$$= \tfrac{3}{2}l$$

$$\boxed{S_\infty = \frac{a}{1-r},\ -1 < r < 1}$$

$$a = \tfrac{1}{2},\ r = \tfrac{1}{2}$$
$$S_\infty = \frac{\tfrac{1}{2}}{1-\tfrac{1}{2}} = \frac{\tfrac{1}{2}}{\tfrac{1}{2}} = 1$$

**4 (b) (iv)**

$$\sum_{k=1}^{n} a_k > (0.99)\sum_{k=1}^{\infty} a_k$$

$$\sum_{k=1}^{n} a_k = \tfrac{3}{2}l\sum_{k=1}^{n}(\tfrac{1}{2})^k$$

Summing formula: $\boxed{S_n = \dfrac{a(1-r^n)}{(1-r)}}$

$$= \tfrac{3}{2}l[(\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots\ldots + (\tfrac{1}{2})^n] \longrightarrow (\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots\ldots + (\tfrac{1}{2})^n$$
$$= \tfrac{3}{2}l[1 - (\tfrac{1}{2})^n]$$

$$a = \tfrac{1}{2},\ r = \tfrac{1}{2}$$
$$S_n = \frac{\tfrac{1}{2}(1-(\tfrac{1}{2})^n)}{1-\tfrac{1}{2}} = \frac{\tfrac{1}{2}(1-(\tfrac{1}{2})^n)}{\tfrac{1}{2}} = 1 - (\tfrac{1}{2})^n$$

Solve the equality:

$$\sum_{k=1}^{n} a_k = (0.99)\sum_{k=1}^{\infty} a_k$$

$$\tfrac{3}{2}l[1-(\tfrac{1}{2})^n] = (0.99)\tfrac{3}{2}l$$

$$1-(\tfrac{1}{2})^n = 0.99$$

$$0.01 = (\tfrac{1}{2})^n$$

$$\log_{10}(0.01) = n\log_{10}(0.5)$$

$$\therefore n = \frac{\log_{10}(0.01)}{\log_{10}(0.5)} = 6.64$$

Therefore, the least positive integer $n$ for which the statement holds is $n = 7$.

**5.** **(a)** Find the coefficient of $x^8$ in the expansion of $(x^2-1)^{10}$.

**(b)** **(i)** Solve the equation:

$$\log_2 x - \log_2(x-1) = 4\log_4 2.$$

**(ii)** Solve the equation:

$$3^{2x+1} - 17(3^x) - 6 = 0.$$

Give your answer correct to two decimal places.

**(c)** Prove by induction that 9 is a factor of $5^{2n+1} + 2^{4n+2}$, for all $n \in \mathbb{N}$.

**SOLUTION**

**5 (a)**

$$u_{r+1} = \binom{10}{r}(x^2)^{10-r}(-1)^r$$

$$\boxed{u_{r+1} = {}^{n}C_r(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r}$$

$$= \binom{10}{r}(-1)^r x^{20-2r}$$

Equate the powers and solve for $r$: $20-2r = 8$

$$12 = 2r$$

$$\therefore r = 6$$

$$r = 6: u_7 = \binom{10}{6}(-1)^6 x^{20-2(6)} = \binom{10}{6}x^8 = 210x^8$$

Coefficient: 210

**5 (b) (i)**

Get out of logs by hooshing, i.e. hoosh $a$ under the $y$ and rub out the log.

$$\log_a x = y \Leftrightarrow x = a^y$$

Log Statement   *Hooshing*   Power Statement

$$\log_4 2 = y \Rightarrow 4^y = 2$$

$$(2^2)^y = 2^1$$

$$2^{2y} = 2^1$$

$$\therefore 2y = 1 \Rightarrow y = \tfrac{1}{2}$$

$$\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$$

$$\log_2 x - \log_2(x-1) = 4\log_4 2$$

$$\log_2\left(\frac{x}{x-1}\right) = 4(\tfrac{1}{2})$$

$$\log_2\left(\frac{x}{x-1}\right) = 2$$

$$\therefore \left(\frac{x}{x-1}\right) = 2^2 = 4$$

$$x = 4(x-1)$$

$$x = 4x - 4$$

$$4 = 3x$$

$$\therefore x = \tfrac{4}{3}$$

**5 (b) (ii)**

$$3^{2x+1} - 17(3^x) - 6 = 0$$

$$(3^x)^2(3^1) - 17(3^x) - 6 = 0$$

$$3(3^x)^2 - 17(3^x) - 6 = 0$$

Let $u = 3^x$

$$3u^2 - 17u - 6 = 0$$

$$(3u+1)(u-6) = 0$$

$$\therefore u = -\tfrac{1}{3},\ 6$$

| | |
|---|---|
| $3^x = -\tfrac{1}{3}$ | $3^x = 6$ |
| No solutions | $x\log_{10} 3 = \log_{10} 6$ |
| | $\therefore x = \dfrac{\log_{10} 6}{\log_{10} 3} = 1.63$ |

**5 (c)**

**STEPS**

**1**. Prove result is true for some starting value of $n \in \mathbf{N}_0$.

**2**. Assuming result is true for $n = k$.

**3**. Prove result is true for $n = (k+1)$.

**1**. Prove for $n = 1$:

$$5^{2(1)+1} + 2^{4(1)+2}$$
$$= 5^3 + 2^6$$
$$= 125 + 64$$
$$= 189$$
$$\frac{189}{9} = 21$$

**2**. Assume for $n = k$:

$$5^{2k+1} + 2^{4k+2} = 9m,\ m \in \mathbb{N}_0$$
$$\Rightarrow 5^{2k+1} = 9m - 2^{4k+2}$$

**3**. Prove for $n = (k + 1)$:

$$5^{2(k+1)+1} + 2^{4(k+1)+2}$$
$$= 5^{2k+3} + 2^{4k+6}$$
$$= 5^2(5^{2k+1}) + 2^{4k+6}$$
$$= 25(9m - 2^{4k+2}) + 2^{4k+6}$$
$$= 25 \times 9m - 25 \times 2^{4k+2} + 2^{4k+6}$$
$$= 25 \times 9m - 2^{4k+2}[25 - 2^4]$$
$$= 25 \times 9m - 2^{4k+2}[25 - 16]$$
$$= 25 \times 9m - 2^{4k+2} \times 9$$
$$= 9[25 \times m - 2^{4k+2}]$$
$$= 9a,\ a \in \mathbb{N}_0$$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.