SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2010

4 (a) Write the recurring decimal 0·474747….. as an infinite geometric series and hence as a fraction.

(b) In an arithmetic sequence, the fifth term is –18 and the tenth term is 12.

(i) Find the first term and the common difference.

(ii) Find the sum of the first fifteen terms of the sequence.

(c) (i) Show that $(r+1)^3-(r-1)^3=6r^2+2$.

(ii) Hence, or otherwise, prove that $\sum_{r=1}^{n} r^2=\dfrac{n(n+1)(2n+1)}{6}$.

(iii) Find $\sum_{r=11}^{30}(3r^2+1)$.

SOLUTION

4 (a)

$$0.474747.....=\frac{47}{100}+\frac{47}{10000}+\frac{47}{1000000}+....$$
$$=\frac{47}{100}\left[1+\frac{1}{100}+\frac{1}{10000}+.....\right]$$
$$=\frac{47}{100}\left[\frac{100}{99}\right]=\frac{47}{99}$$

$$\boxed{S_\infty=\frac{a}{1-r},\ -1<r<1}$$

$a=1, r=\frac{1}{100}$

$$S_\infty=\frac{1}{1-\frac{1}{100}}=\frac{1}{\frac{99}{100}}=\frac{100}{99}$$

4 (b) (i)

> The fifty-sixth term of an arithmetic sequence: $u_{56}=a+55d$

$u_5=-18\Rightarrow a+4d=-18....(\mathbf{1})$

$u_{10}=12\Rightarrow a+9d=12....(\mathbf{2})$

$$\begin{aligned} a+4d&=-18....(\mathbf{1})\\ -a-9d&=-12....(\mathbf{2})(\times -1)\\ \hline -5d&=-30\Rightarrow d=6 \end{aligned}$$

Replace d by its value in Eqn. (**1**):

$a+4(6)=-18$

$a+24=-18$

$a=-42$

4 (b) (ii)

$$\boxed{S_n=\tfrac{n}{2}[2a+(n-1)d]}$$

$$S_{15}=\tfrac{15}{2}[2(-42)+(15-1)6]$$
$$=\tfrac{15}{2}[-84+14(6)]$$
$$=\tfrac{15}{2}[-84+84]$$
$$=\tfrac{15}{2}[0]=0$$

4 (c) (i)

$$(r+1)^3-(r-1)^3$$
$$=r^3+3r^2+3r+1-(r^3-3r^2+3r-1)$$
$$=r^3+3r^2+3r+1-r^3+3r^2-3r+1$$
$$=6r^2+2$$

4 (c) (ii)

1. Prove this statement is true for $n = 1$.
2. Assume it is true for $n = k$.

$$\sum_{r=1}^{k} k^2 = \underline{\{1^2+2^2+\ldots+k^2\}=\tfrac{k}{6}(k+1)(2k+1)}$$

3. Prove for $n = k + 1$.

$$\sum_{r=1}^{k+1}(k+1)^2 = \{\underline{1^2+2^2+\ldots+k^2}\}+(k+1)^2 = \tfrac{k+1}{6}(k+2)(2k+3)$$

Use the result in Step 2 to prove step 3.

$$\Rightarrow \tfrac{k}{6}(k+1)(2k+1)+(k+1)^2 = \tfrac{k+1}{6}(k+2)(2k+3)$$

Show this result is true and your proof is complete.

4 (c) (iii)

$$S_n=\sum_{r=1}^{n}(3r^2+1)$$
$$=3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}1$$
$$=3\left(\frac{n(n+1)(2n+1)}{6}\right)+n$$
$$=\frac{n(n+1)(2n+1)}{2}+n$$

$$S_{30}-S_{10}=\sum_{r=11}^{30}(3r^2+1)$$
$$=\left[\frac{30(30+1)(2(30)+1)}{2}+(30)\right]-\left[\frac{10(10+1)(2(10)+1)}{2}+(10)\right]$$
$$=\left[\frac{30(31)(61)}{2}+(30)\right]-\left[\frac{10(11)(21)}{2}+(10)\right]=27,230$$

5 (a) Solve the equation: $\log_2(x+6)-\log_2(x+2)=1$.

(b) Use induction to prove that

$$2+(2\times 3)+(2\times 3^2)+.............+(2\times 3^{n-1})=3^n-1,$$

where n is a positive integer.

(c) (i) Expand $\left(x+\dfrac{1}{x}\right)^2$ and $\left(x+\dfrac{1}{x}\right)^4$.

(ii) Hence, or otherwise, find the value of $x^4+\dfrac{1}{x^4}$, given that $x+\dfrac{1}{x}=3$.

Solution

5 (a)

$$\log_a M-\log_a N=\log_a\left(\frac{M}{N}\right)$$

$$\log_2(x+6)-\log_2(x+2)=1$$

$$\log_2\left(\frac{x+6}{x+2}\right)=1$$

$$\frac{x+6}{x+2}=2^1$$

$$x+6=2(x+2)$$

$$x+6=2x+4$$

$$2=x$$

5 (b)

Steps

1. Prove result is true for some starting value of $n\in\mathbf{N}_0$.

2. Assuming result is true for n = k.

3. Prove result is true for n = (k +1).

1. Prove for n = 1:

$$2\times 3^{1-1}=3^1-1$$

$$2\times 3^0=3-1$$

$$2\times 1=2$$

$$2=2$$

2. Assume it is true for n = k:

$$2+(2\times 3)+(2\times 3^2)+.........+(2\times 3^{k-1})=3^k-1$$

$$\Rightarrow 2+(2\times 3)+(2\times 3^2)+.........+(2\times 3^{k-1})+1=3^k$$

3. Prove it is true for n = k + 1:

$$3^{k+1}-1=2+(2\times 3)+(2\times 3^2)+.........+(2\times 3^k)$$

$$3^{k+1}-1$$

$$=3\times 3^k-1$$

$$=3(2+(2\times 3)+(2\times 3^2)+.........+(2\times 3^{k-1})+1)-1$$

$$=2\times 3+2\times 3^2+2\times 3^3+........+2\times 3^k+3-1$$

$$=2+2\times 3+2\times 3^2+2\times 3^3+........+2\times 3^k$$

5 (c) (i)

$$\left(x+\frac{1}{x}\right)^2 = x^2 + 2x \times \frac{1}{x} + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2}$$

$$\begin{aligned}\left(x+\frac{1}{x}\right)^4 &= \left(x+\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^2 \\ &= \left(x^2+2+\frac{1}{x^2}\right)\left(x^2+2+\frac{1}{x^2}\right) \\ &= x^4 + 2x^2 + 1 + 2x^2 + 4 + \frac{2}{x^2} + 1 + \frac{2}{x^2} + \frac{1}{x^4} \\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\end{aligned}$$

5 (c) (ii)

$$\left(x+\frac{1}{x}\right)^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

$$(3)^4 = x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

$$81 - 6 - 4x^2 + \frac{4}{x^2} = x^4 + \frac{1}{x^4}$$

$$75 - 4\left(x^2 + \frac{1}{x^2}\right) = x^4 + \frac{1}{x^4}$$

$$\left(x+\frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$$

$$(3)^2 = x^2 + 2 + \frac{1}{x^2}$$

$$9 - 2 = x^2 + \frac{1}{x^2}$$

$$7 = x^2 + \frac{1}{x^2}$$

$$x^4 + \frac{1}{x^4} = 75 - 4(7) = 75 - 28 = 47$$