# SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

## 2009

4 (a) Three consecutive terms of an arithmetic series are $4x + 11$, $2x + 11$ and $3x + 17$. Find the value of $x$.

(b) (i) Show that $\dfrac{2}{r^2-1} = \dfrac{1}{r-1} - \dfrac{1}{r+1}$, where $r \neq \pm 1$.

(ii) Hence, find $\displaystyle\sum_{r=2}^{n} \frac{2}{r^2-1}$.

(iii) Hence, evaluate $\displaystyle\sum_{r=2}^{\infty} \frac{2}{r^2-1}$.

(c) A finite geometric sequence has first term $a$ and common ratio $r$.
The sequence has $2m + 1$ terms, where $m \in \mathbf{N}$.

(i) Write down the last term, in terms of $a$, $r$, and $m$.

(ii) Write down the middle term, in terms of $a$, $r$, and $m$.

(iii) Show that the product of all the terms of the sequence is equal to the middle term raised to the power of the number of terms.

**SOLUTION**

**4 (a)**

$(2x+11)-(4x+11)=(3x+17)-(2x+11)$

$2x+11-4x-11=3x+17-2x-11$

$-2x=x+6$

$-3x=6$

$x=-2$

[Subtracting consecutive terms of an arithmetic sequence gives the same number (the common difference $d$)].

**4 (b) (i)**

**LHS**

$\dfrac{2}{r^2-1}$

**RHS**

$\dfrac{1}{r-1}-\dfrac{1}{r+1}$

$=\dfrac{1(r+1)-1(r-1)}{(r-1)(r+1)}$

$=\dfrac{r+1-r+1}{(r-1)(r+1)}$

$=\dfrac{2}{r^2-1}$

**4 (b) (ii)**

$$S_n = \sum_{r=2}^{n} \frac{2}{r^2-1} = \sum_{r=2}^{n}\left[\frac{1}{r-1} - \frac{1}{r+1}\right]$$

$$u_r = \frac{2}{r^2-1} = \frac{1}{r-1} - \frac{1}{r+1}$$

$$u_2 = \frac{1}{1} - \frac{1}{3}$$

$$u_3 = \frac{1}{2} - \frac{1}{4}$$

$$u_4 = \frac{1}{3} - \frac{1}{5}$$

⋮

$$u_{n-1} = \frac{1}{n-2} - \frac{1}{n}$$

$$u_n = \frac{1}{n-1} - \frac{1}{n+1}$$

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$$\therefore S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

**4 (b) (iii)**

$$\therefore S_\infty = \frac{3}{2}$$

**4 (c) (i)**

**Geometric sequence**

First term: $a$

Common ratio: $r$

No. of terms $n = 2m + 1$

General term: $u_n = ar^{n-1}$

$$a,\ ar,\ ar^2, \ldots\ldots\ldots\ldots, ar^{n-1}$$

$$= a,\ ar,\ ar^2, \ldots\ldots\ldots\ldots, ar^{(2m+1)-1}$$

$$= a,\ ar,\ ar^2, \ldots\ldots\ldots\ldots, ar^{2m}$$

**4 (c) (ii)**

Take an example: $m = 4$, $n = 2m + 1 = 9$

$$\begin{array}{ccccccccc} a, & ar, & ar^2, & ar^3, & ar^4, & ar^5, & ar^6, & ar^7, & ar^8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 \end{array}$$

Middle term: $ar^m$

**4 (c) (iii)**

$a \times ar \times ar^2 \times ar^3 \times ........\times ar^{2m}$

$= a^{2m+1} r^{1+2+3+.....+2m} \leftarrow *$

$= a^{2m+1} r^{m(2m+1)}$

$(ar^m)^{2m+1}$

$= a^{2m+1} r^{m(2m+1)}$

* The power of $r$ is an arithmetic series which sums to $m(2m + 1)$. The method for arriving at this result is shown below.

Arithmetic sequence: 1, 2, 3, ..., $2m$  Summing formula: $S_n = \frac{n}{2}[2a + (n-1)d]$

$a = 1$, $r = 1$, $n = 2m$

$S_n = \frac{2m}{2}[2(1) + (2m-1)(1)]$

$= m[2 + 2m - 1]$

$= m(2m+1)$

5 (a) Solve for $x$: $x - 2 = \sqrt{3x-2}$.

(b) Prove by induction that, for all positive integers $n$, 5 is a factor of $n^5 - n$.

(c) Solve the simultaneous equations

$$\log_3 x + \log_3 y = 2$$
$$\log_3(2y-3) - 2\log_9 x = 1.$$

**SOLUTION**

**5 (a)**

$x - 2 = \sqrt{3x-2}$

$(x-2)^2 = 3x - 2$

$x^2 - 4x + 4 = 3x - 2$

$x^2 - 7x + 6 = 0$

$(x-1)(x-6) = 0$

$\therefore x = 1, 6$

Check each solution:

$x = 1$:

| | |
|---|---|
| $x-2$ | $\sqrt{3x-2}$ |
| $(1)-2$ | $\sqrt{3(1)-2}$ |
| $-1$ | $\sqrt{3-2}$ |
| | $\sqrt{1}$ |
| | $1$ |

$x = 6$:

| | |
|---|---|
| $x-2$ | $\sqrt{3(6)-2}$ |
| $(6)-2$ | $\sqrt{18-2}$ |
| $4$ | $\sqrt{16}$ |
| | $4$ |

**ANS**: $x = 6$

**5 (b)**

**Steps**
**1**. Prove result is true for some starting value of $n \in \mathbf{N}_0$.
**2**. Assuming result is true for $n = k$.
**3**. Prove result is true for $n = (k+1)$.

**1**. Prove it is true for $n = 1$.

$n = 1: (1)^5 - (1) = 1 - 1 = 0$

0 is divisible by 5. Therefore, it is true for $n = 1$.

**2**. Assume it is true for $n = k$:

$n = k: k^5 - k = 5a,\ a \in \mathbf{N}$

$\Rightarrow k^5 = 5a + k$

**3**. Prove it is true for $n = k + 1$:

$$(x+y)^n = \binom{n}{0}(x)^n(y)^0 + \binom{n}{1}(x)^{n-1}(y)^1 + \binom{n}{2}(x)^{n-2}(y)^2 \ldots\ldots$$

$(k+1)^5 - (k+1)$

$= \binom{5}{0}k^5 + \binom{5}{1}k^4 + \binom{5}{2}k^3 + \binom{5}{3}k^2 + \binom{5}{4}k + \binom{5}{5} - k - 1$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$ [Replace $k^5$ by the result from Step **2**.]

$= 5a + k + 5k^4 + 10k^3 + 10k^2 + 5k - k$

$= 5a + 5k^4 + 10k^3 + 10k^2 + 5k$

$= 5(a + k^4 + 2k^3 + 2k^2 + k) = 5m,\ m \in \mathbf{N}$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.

**5 (c)**

Change all logs to base 3:

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{2}$$

$$\log_a M = \frac{\log_b M}{\log_b a}$$

$$\begin{aligned}&\log_3 x + \log_3 y = 2\\ &\log_3(2y-3) - 2\log_9 x = 1\end{aligned} \rightarrow \begin{aligned}&\log_3 x + \log_3 y = 2\\ &\log_3(2y-3) - 2(\tfrac{1}{2}\log_3 x) = 1\end{aligned}$$

$\log_3 x + \log_3 y = 2 \ldots\ldots(\mathbf{1})$

$-\log_3 x + \log_3(2y-3) = 1 \ldots(\mathbf{2})$

$\log_3 y + \log_3(2y-3) = 3$

$\log_3 y(2y-3) = 3$

$y(2y-3) = 3^3$

$2y^2 - 3y = 27$

$2y^2 - 3y - 27 = 0$

$(2y-9)(y+3) = 0$

$y = \frac{9}{2}, -3$

Ignore the negative value of $y$ as the log of negative values is not allowed.

Substitute this value of $y$ into Eqn. (**2**):

$-\log_3 x + \log_3(2(\tfrac{9}{2}) - 3) = 1$

$-\log_3 x + \log_3(9 - 3) = 1$

$-\log_3 x + \log_3(6) = 1$

$\log_3(\tfrac{6}{x}) = 1 \Rightarrow \tfrac{6}{x} = 3^1$

$\therefore x = 2$