

**SEQUENCES & SERIES (Q 4 & 5, PAPER 1)**

**2008**

- 4 (a)  $2 + \frac{2}{3} + \frac{2}{9} + \dots$  is a geometric series.  
Find the sum to infinity of the series.
- (b) Given that  $u_n = 2(-\frac{1}{2})^n - 2$  for all  $n \in \mathbf{N}$ ,
- (i) write down  $u_{n+1}$  and  $u_{n+2}$
- (ii) show that  $2u_{n+2} - u_{n+1} - u_n = 0$ .
- (c) (i) Write down an expression in  $n$  for the sum  $1 + 2 + 3 + \dots + n$   
and an expression in  $n$  for the sum  $1^2 + 2^2 + 3^2 + \dots + n^2$ .
- (ii) Find, in terms of  $n$ , the sum  $\sum_{r=1}^n (6r^2 + 2r + 5 + 2^r)$ .

**SOLUTION**

**4 (a)**

$$a = 2, r = \frac{1}{3}$$

$$S_\infty = \frac{2}{1 - \frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

$$S_\infty = \frac{a}{1-r}, -1 < r < 1 \dots\dots \textcircled{6}$$

**4 (b) (i)**

$$u_n = 2(-\frac{1}{2})^n - 2$$

$$\therefore u_{n+1} = 2(-\frac{1}{2})^{n+1} - 2$$

$$\therefore u_{n+2} = 2(-\frac{1}{2})^{n+2} - 2$$

**4 (b) (i)**

$$\begin{aligned} & 2u_{n+2} - u_{n+1} - u_n \\ &= 2[2(-\frac{1}{2})^{n+2} - 2] - [2(-\frac{1}{2})^{n+1} - 2] - [2(-\frac{1}{2})^n - 2] \\ &= 4(-\frac{1}{2})^{n+2} - \cancel{4} - 2(-\frac{1}{2})^{n+1} + \cancel{2} - 2(-\frac{1}{2})^n + \cancel{2} \\ &= (-\frac{1}{2})^n [4(-\frac{1}{2})^2 - 2(-\frac{1}{2})^1 - 2] \\ &= (-\frac{1}{2})^n [4(\frac{1}{4}) - 2(-\frac{1}{2}) - 2] \\ &= (-\frac{1}{2})^n [1 + 1 - 2] = (-\frac{1}{2})^n [0] \\ &= 0 \end{aligned}$$

**4 (c) (i)**

$$\sum_{r=1}^n r = S_n = 1 + 2 + \dots + n = \frac{n}{2}(n+1) \dots\dots \textcircled{7}$$

$$\sum_{r=1}^n r^2 = S_n = 1^2 + 2^2 + \dots + n^2 = \frac{n}{6}(n+1)(2n+1) \dots\dots \textcircled{8}$$

**4 (c) (ii)**

$$\sum_{r=1}^n (6r^2 + 2r + 5 + 2^r)$$

$$= 6 \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r + 5 \sum_{r=1}^n 1 + \sum_{r=1}^n 2^r$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots + 1 = n$$

$$= 6\left(\frac{n}{6}\right)(n+1)(2n+1) + 2\left(\frac{n}{2}\right)(n+1) + 5n + \{2^1 + 2^2 + 2^3 + \dots + 2^n\}$$

$$= n(n+1)(2n+1) + n(n+1) + 5n + \{2^1 + 2^2 + 2^3 + \dots + 2^n\}$$

$$S_n = \frac{a(1-r^n)}{(1-r)} \dots \dots \dots \mathbf{5}$$

This is a geometric series with  $a = 2$ ,  $r = 2$

$$S_n = \frac{2(1-2^n)}{1-2} = 2(2^n - 1)$$

$$\therefore \sum_{r=1}^n (6r^2 + 2r + 5 + 2^r) = n(n+1)(2n+1) + n(n+1) + 5n + 2(2^n - 1)$$

5 (a) Find the range of values of  $x$  which satisfy the inequality

$$x^2 - 3x - 10 \leq 0.$$

(b) (i) Solve the equation

$$2^{x^2} = 8^{2x+9}.$$

(ii) Solve the equation

$$\log_e(2x+3) + \log_e(x-2) = 2\log_e(x+4).$$

(c) Show that there are no natural numbers  $n$  and  $r$  for which

$\binom{n}{r-1}$ ,  $\binom{n}{r}$  and  $\binom{n}{r+1}$  are consecutive terms in a geometric sequence.

**SOLUTION**

**5 (a)**

[A] **QUADRATICS:**  $ax^2 + bx + c \leq 0$

**STEPS**

1. Get all terms on one side and zero on the other side.
2. Solve the corresponding equation to get the roots  $\alpha$ ,  $\beta$ .
3. Carry out the region test. Use the roots in ascending order to form regions:  $\leftarrow \alpha \leftrightarrow \beta \rightarrow$  Choose a nice number in each region to test the inequality using the **test box**.
4. Based on the region test write down the solutions.

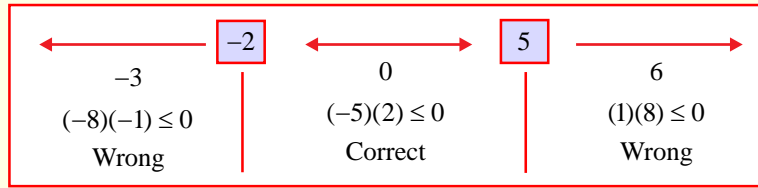
Solve  $x^2 - 3x - 10 = 0$ .

$$x^2 - 3x - 10 = 0$$

$$\Rightarrow (x-5)(x+2) = 0$$

$$\therefore x = -2, 5$$

Do the region test:



**Test box:**  $(x - 5)(x + 2) \leq 0$

**ANS:**  $-2 \leq x \leq 5$

**5 (b) (i)**

**STEPS: SOLVING SIMPLE EXPONENTIALS**

1. Tidy up the algebra using the power rules shown.
2. *EITHER*, express everything in the same base, equate the powers and solve for  $x$ ,  
*OR* take the common log of both sides if you cannot get the same base.

$$2^{x^2} = 8^{2x+9}$$

$$\Rightarrow 2^{x^2} = (2^3)^{2x+9} \text{ [Use Power Rule No. 5]}$$

$$\Rightarrow 2^{x^2} = 2^{6x+27}$$

$$\therefore x^2 = 6x + 27$$

$$\Rightarrow x^2 - 6x - 27 = 0$$

$$\Rightarrow (x - 9)(x + 3) = 0$$

$$\therefore x = -3, 9$$

**POWER RULES**

- |                                |                                 |
|--------------------------------|---------------------------------|
| 1. $a^m \times a^n = a^{m+n}$  | 4. $a^{-n} = \frac{1}{a^n}$     |
| 2. $\frac{a^m}{a^n} = a^{m-n}$ | 5. $(a^m)^n = a^{mn}$           |
| 3. $a^0 = 1$                   | 6. $\sqrt{a} = a^{\frac{1}{2}}$ |

**5 (b) (ii)**

**LOG RULES**

- |  |   |
|--|---|
| 1. $\log_a M + \log_a N = \log_a (MN)$                     | 4. $\log_a M = \frac{\log_b M}{\log_b a}$ [Used to change base] |
| 2. $\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)$ | 5. $\log_a 1 = 0$ and $\log_a a = 1$                            |
| 3. $N \log_a M = \log_a (M^N)$                             | 6. $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$  |

$$\log_e (2x + 3) + \log_e (x - 2) = 2 \log_e (x + 4)$$

$$\Rightarrow \log_e (2x + 3) + \log_e (x - 2) - 2 \log_e (x + 4) = 0 \text{ [Use Log rules 1, 2 and 3.]}$$

$$\Rightarrow \log_e \left[ \frac{(2x + 3)(x - 2)}{(x + 4)^2} \right] = 0$$

$$\Rightarrow \left[ \frac{(2x + 3)(x - 2)}{(x + 4)^2} \right] = e^0 = 1$$

$$\Rightarrow (2x + 3)(x - 2) = (x + 4)^2$$

$$\Rightarrow 2x^2 - x - 6 = x^2 + 8x + 16$$

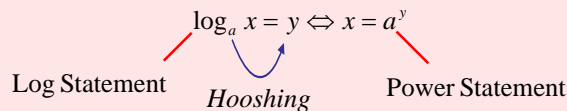
$$\Rightarrow x^2 - 9x - 22 = 0$$

$$\Rightarrow (x - 11)(x + 2) = 0$$

$$\therefore x = 11, -2$$

$x = 11$  is the only solution as the other solution will give you the log of a negative number which is not allowed.

Get out of logs by hooshing, i.e. hoosh  $a$  under the  $y$  and rub out the log.



**5 (c)**

A sequence is geometric if when you divide two successive terms you obtain the common ratio.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\therefore \frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{\binom{n}{r+1}}{\binom{n}{r}}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{r!(n-r)!}}$$

$$\Rightarrow \frac{\cancel{n!}}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{\cancel{n!}} = \frac{\cancel{n!}}{(r+1)!(n-r-1)!} \times \frac{r!(n-r)!}{\cancel{n!}}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{n-r}{r+1}$$

$$\Rightarrow (r+1)(n-r+1) = r(n-r)$$

$$\Rightarrow rn - r^2 + r + n - r + 1 = rn - r^2$$

$$\therefore n = -1$$

This is not a natural number. Therefore, there are no natural numbers for which the statement is true.