SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2008

4 (a) $2+\frac{2}{3}+\frac{2}{9}+......$ is a geometric series.
Find the sum to infinity of the series.

(b) Given that $u_n = 2(-\frac{1}{2})^n - 2$ for all $n \in \mathbf{N}$,

(i) write down u_{n+1} and u_{n+2}

(ii) show that $2u_{n+2} - u_{n+1} - u_n = 0$.

(c) (i) Write down an expression in n for the sum 1 + 2 + 3 +........+ n and an expression in n for the sum $1^2+2^2+3^2+......+n^2$.

(ii) Find, in terms of n, the sum $\sum_{r=1}^{n}(6r^2+2r+5+2^r)$.

SOLUTION

4 (a)

$a=2,\ r=\frac{1}{3}$

$S_\infty = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$

$S_\infty = \frac{a}{1-r},\ -1<r<1$ 6

4 (b) (i)

$u_n = 2(-\frac{1}{2})^n - 2$

$\therefore u_{n+1} = 2(-\frac{1}{2})^{n+1} - 2$

$\therefore u_{n+2} = 2(-\frac{1}{2})^{n+2} - 2$

4 (b) (i)

$2u_{n+2} - u_{n+1} - u_n$

$= 2[2(-\frac{1}{2})^{n+2} - 2] - [2(-\frac{1}{2})^{n+1} - 2] - [2(-\frac{1}{2})^n - 2]$

$= 4(-\frac{1}{2})^{n+2} - \cancel{4} - 2(-\frac{1}{2})^{n+1} + \cancel{2} - 2(-\frac{1}{2})^n + \cancel{2}$

$= (-\frac{1}{2})^n[4(-\frac{1}{2})^2 - 2(-\frac{1}{2})^1 - 2]$

$= (-\frac{1}{2})^n[4(\frac{1}{4}) - 2(-\frac{1}{2}) - 2]$

$= (-\frac{1}{2})^n[1+1-2] = (-\frac{1}{2})^n[0]$

$= 0$

4 (c) (i)

$\sum_{r=1}^{n} r = S_n = 1+2+....+n = \frac{n}{2}(n+1)$ 7

$\sum_{r=1}^{n} r^2 = S_n = 1^2+2^2+......+n^2 = \frac{n}{6}(n+1)(2n+1)$ 8

4 (c) (ii)

$$\sum_{r=1}^{n}(6r^2+2r+5+2^r)$$

$$=6\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r+5\sum_{r=1}^{n}1+\sum_{r=1}^{n}2^r$$

$$\sum_{r=1}^{n}1=1+1+.....+1=n$$

$$=6(\tfrac{n}{6})(n+1)(2n+1)+2(\tfrac{n}{2})(n+1)+5n+\{2^1+2^2+2^3+.....+2^n\}$$

$$=n(n+1)(2n+1)+n(n+1)+5n+\{2^1+2^2+2^3+.....+2^n\}$$

$$S_n=\frac{a(1-r^n)}{(1-r)} \quad .... \quad 5$$

This is a geometric series with $a = 2$, $r = 2$

$$S_n=\frac{2(1-2^n)}{1-2}=2(2^n-1)$$

$$\therefore \sum_{r=1}^{n}(6r^2+2r+5+2^r)=n(n+1)(2n+1)+n(n+1)+5n+2(2^n-1)$$

5 (a) Find the range of values of x which satisfy the inequality

$$x^2-3x-10\le 0.$$

(b) (i) Solve the equation

$$2^{x^2}=8^{2x+9}.$$

(ii) Solve the equation

$$\log_e(2x+3)+\log_e(x-2)=2\log_e(x+4).$$

(c) Show that there are no natural numbers n and r for which

$\binom{n}{r-1}$, $\binom{n}{r}$ and $\binom{n}{r+1}$ are consecutive terms in a geometric sequence.

Solution

5 (a)

[A] **Quadratics:** $ax^2+bx+c\le 0$

Steps
1. Get all terms on one side and zero on the other side.
2. Solve the corresponding equation to get the roots $\alpha,\ \beta$.
3. Carry out the region test. Use the roots in ascending order to form regions: $\leftarrow \alpha \leftrightarrow \beta \rightarrow$ Choose a nice number in each region to test the inequality using the **test box**.
4. Based on the region test write down the solutions.

Solve $x^2-3x-10=0$.

$x^2-3x-10=0$

$\Rightarrow (x-5)(x+2)=0$

$\therefore x=-2,\ 5$

Do the region test:

$x \le -2$	-2	$-2 \le x \le 5$	5	$x \ge 5$
-3		0		6
$(-8)(-1) \le 0$		$(-5)(2) \le 0$		$(1)(8) \le 0$
Wrong		Correct		Wrong

Test box: $(x-5)(x+2) \le 0$

Ans: $-2 \le x \le 5$

5 (b) (i)

> **Steps: Solving Simple Exponentials**
> **1.** Tidy up the algebra using the power rules shown.
> **2.** *Either*, express everything in the same base, equate the powers and solve for x,
> *or* take the common log of both sides if you cannot get the same base.

$2^{x^2} = 8^{2x+9}$

$\Rightarrow 2^{x^2} = (2^3)^{2x+9}$ [Use Power Rule No. 5]

$\Rightarrow 2^{x^2} = 2^{6x+27}$

$\therefore x^2 = 6x + 27$

$\Rightarrow x^2 - 6x - 27 = 0$

$\Rightarrow (x-9)(x+3) = 0$

$\therefore x = -3,\ 9$

> **Power Rules**
> 1. $a^m \times a^n = a^{m+n}$
> 2. $\dfrac{a^m}{a^n} = a^{m-n}$
> 3. $a^0 = 1$
> 4. $a^{-n} = \dfrac{1}{a^n}$
> 5. $(a^m)^n = a^{mn}$
> 6. $\sqrt{a} = a^{\frac{1}{2}}$

5 (b) (ii)

> **Log Rules**
> 1. $\log_a M + \log_a N = \log_a (MN)$
> 2. $\log_a M - \log_a N = \log_a \left(\dfrac{M}{N}\right)$
> 3. $N \log_a M = \log_a (M^N)$
> 4. $\log_a M = \dfrac{\log_b M}{\log_b a}$ [Used to change base]
> 5. $\log_a 1 = 0$ and $\log_a a = 1$
> 6. $\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$

$\log_e (2x+3) + \log_e (x-2) = 2\log_e (x+4)$

$\Rightarrow \log_e (2x+3) + \log_e (x-2) - 2\log_e (x+4) = 0$ [Use Log rules 1, 2 and 3.]

$\Rightarrow \log_e \left[\dfrac{(2x+3)(x-2)}{(x+4)^2}\right] = 0$

$\Rightarrow \left[\dfrac{(2x+3)(x-2)}{(x+4)^2}\right] = e^0 = 1$

> Get out of logs by hooshing, i.e. hoosh a under the y and rub out the log.
> $\log_a x = y \Leftrightarrow x = a^y$
> Log Statement — *Hooshing* — Power Statement

$\Rightarrow (2x+3)(x-2) = (x+4)^2$

$\Rightarrow 2x^2 - x - 6 = x^2 + 8x + 16$

$\Rightarrow x^2 - 9x - 22 = 0$

$\Rightarrow (x-11)(x+2) = 0$

$\therefore x = 11,\ -2$

$x = 11$ is the only solution as the other solution will give you the log of a negative number which is not allowed.

5 (c)

A sequence is geometric if when you divide two successive terms you obtain the common ratio.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\therefore \frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{\binom{n}{r+1}}{\binom{n}{r}}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{r!(n-r)!}}$$

$$\Rightarrow \frac{\cancel{n}!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{\cancel{n}!} = \frac{\cancel{n}!}{(r+1)!(n-r-1)!} \times \frac{r!(n-r)!}{\cancel{n}!}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{n-r}{r+1}$$

$$\Rightarrow (r+1)(n-r+1) = r(n-r)$$

$$\Rightarrow rn - r^2 + r + n - r + 1 = rn - r^2$$

$$\therefore n = -1$$

This is not a natural number. Therefore, there are no natural numbers for which the statement is true.