SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

2000

4 (a) The first three terms of a geometric sequence are

$$2x-4, x+1, x-3.$$

Find the two possible values of x.

4 (b) Given that $u_n = \frac{1}{2}(4^n - 2^n)$ for all integers n, show that

$$u_{n+1} = 2u_n + 4^n.$$

4 (c) (i) Given that $g(x) = 1 + 2x + 3x^2 + 4x^3 \ldots$ where $-1 < x < 1$, show that

$$g(x) = \frac{1}{(1-x)^2}.$$

(ii) $P(n) = u_1 u_2 u_3 u_4 \ldots u_n$ where $u_k = ar^{k-1}$ for $k = 1, 2, 3, \ldots, n$ and $a, r \in \mathbf{R}$.

Write $P(n)$ in the form $a^n r^{f(n)}$ where $f(n)$ is a quadratic expression in n.

SOLUTION

4 (a)

Test for a geometric sequence: $\frac{u_{n+1}}{u_n} = \text{Constant} = r$

$$\frac{x+1}{2x-4} = \frac{x-3}{x+1}$$

$\Rightarrow (x+1)^2 = (2x-4)(x-3)$

$\Rightarrow x^2 + 2x + 1 = 2x^2 - 10x + 12$

$\Rightarrow x^2 - 12x + 11 = 0$

$\Rightarrow (x-1)(x-11) = 0$

$\therefore x = 1, 11$

4 (b)

$u_n = \frac{1}{2}(4^n - 2^n) \Rightarrow u_{n+1} = \frac{1}{2}(4^{n+1} - 2^{n+1})$

LHS	*RHS*
$u_{n+1} = \frac{1}{2}(4^{n+1} - 2^{n+1})$	$2u_n + 4^n$
$= \frac{1}{2}(4 \times 4^n - 2 \times 2^n)$	$= 2[\frac{1}{2}(4^n - 2^n)] + 4^n$
$= 2 \times 4^n - 2^n$	$= 4^n - 2^n + 4^n$
	$= 2 \times 4^n - 2^n$

4 (c) (i)

$$g(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$
$$xg(x) = x + 2x^2 + 3x^3 + \ldots$$
$$g(x) - xg(x) = x + x^2 + x^3 + \ldots$$
$$\Rightarrow (1-x)g(x) = x + x^2 + x^3 + \ldots$$

$x + x^2 + x^3 + \ldots$ is an infinite geometric series.

$a = x,\ r = x$

$$S_\infty = \frac{x}{1-x}$$ $\boxed{S_\infty = \frac{a}{1-r},\ -1 < r < 1}$ ……. **6**

$$\therefore (1-x)g(x) = \frac{x}{1-x}$$
$$\Rightarrow g(x) = \frac{x}{(1-x)^2}$$

4 (c) (ii)

$$P(n) = u_1 u_2 u_3 u_4 \ldots u_n$$
$$\Rightarrow P(n) = (ar^0)(ar^1)(ar^2)(ar^3)\ldots(ar^{n-1})$$
$$\Rightarrow P(n) = a^n(r^{0+1+2+3+\ldots+n-1})$$

The power of r is an arithmetic series with $a = 0$, $d = 1$.

$$S_n = \tfrac{n}{2}[2(0) + (n-1)1]$$ $\boxed{S_n = \tfrac{n}{2}[2a + (n-1)d]}$ ……. **3**
$$\Rightarrow S_n = \tfrac{n}{2}(n-1)$$

$$\therefore P(n) = a^n r^{\frac{n}{2}(n-1)}$$

5 (a) Express the recurring decimal $1.\dot{2}$ in the form $\frac{a}{b}$ where $a, b \in \mathbf{N}$.

5 (b) Prove by induction that $n! > 2^n$, $n \in \mathbf{N}$, $n \geq 4$.

5 (c) (i) Solve for x

$$2\log_9 x = \tfrac{1}{2} + \log_9(5x + 18),\ x > 0.$$

(ii) Solve for x

$$3e^x - 7 + 2e^{-x} = 0.$$

SOLUTION

5 (a)

$$1.\dot{2} = 1.22222\ldots = 1 + \tfrac{2}{10} + \tfrac{2}{100} + \tfrac{2}{1000} + \ldots = 1 + \tfrac{2}{10}[1 + \tfrac{1}{10} + \tfrac{1}{100} + \ldots]$$

Consider the expression in the bracket: $a = 1,\ r = \dfrac{1}{10} \Rightarrow S_\infty = \dfrac{1}{1 - \frac{1}{10}} = \dfrac{1}{\frac{9}{10}} = \dfrac{10}{9}$

$$\therefore 1.\dot{2} = 1 + \tfrac{2}{10}(\tfrac{10}{9}) = 1 + \tfrac{2}{9} = \tfrac{11}{9}$$

5 (b)

> **Steps**
> **1.** Prove result is true for some starting value of $n \in \mathbf{N}_0$.
> **2.** Assuming result is true for $n = k$.
> **3.** Prove result is true for $n = (k+1)$.

1. Prove this statement is true for $n = 4$.

$4! > 2^4 \Rightarrow 24 > 16$. Therefore, it is true for $n = 4$.

2. Assume it is true for $n = k \Rightarrow k! > 2^k$.

3. Prove for $n = k + 1$. Show that $(k+1)! > 2^{k+1}$.

$(k+1)! = \underline{k!}\underline{(k+1)} > \underline{2^k} \times \underline{2}$ [This statement is true as the red part is true and the blue part is true.]

$\underline{k > 4 \Rightarrow k+1 > 5 > 2}$ From Step 2: $\underline{k! > 2^k}$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.

5 (c) (i)

$2\log_9 x = \frac{1}{2} + \log_9(5x+18)$

$\Rightarrow 2\log_9 x - \log_9(5x+18) = \frac{1}{2}$

$\Rightarrow \log_9\left(\frac{x^2}{5x+18}\right) = \frac{1}{2}$ [Using log rules 1 and 3.]

$\Rightarrow \frac{x^2}{5x+18} = 9^{\frac{1}{2}} = 3$

$\Rightarrow x^2 = 3(5x+18)$

$\Rightarrow x^2 = 15x + 54$

$\Rightarrow x^2 - 15x - 54 = 0$

$\Rightarrow (x-18)(x+3) = 0$

$\therefore x = 18$, ~~-3~~ as $x > 0$

> **Log Rules**
> 1. $\log_a M + \log_a N = \log_a(MN)$
> 2. $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$
> 3. $N \log_a M = \log_a(M^N)$

5 (c) (i)

> **Steps: Solving Sums of Exponential Functions**
> **1.** Isolate the common exponential function by putting it in brackets.
> **2.** Make a substitution by putting the object inside the bracket equal to a letter, say u.
> **3.** Solve the resulting quadratic.
> **4.** Find all answers for the original variable.

1. $3e^x - 7 + 2e^{-x} = 0 \Rightarrow 3e^x - 7 + \frac{2}{e^x} = 0$

2. Let $u = e^x \Rightarrow 3u - 7 + \frac{2}{u} = 0$

3. $\therefore 3u^2 - 7u + 2 = 0 \Rightarrow (3u-1)(u-2) = 0$

$\therefore u = \frac{1}{3}, 2$

4.

$e^x = \frac{1}{3}$

$\Rightarrow \ln e^x = \ln(\frac{1}{3})$

$\Rightarrow x = \ln 1 - \ln 3$

$\therefore x = -\ln 3$

$e^x = 2$

$\Rightarrow \ln e^x = \ln 2$

$\therefore x = \ln 2$