1999

4 (a) Solve $\binom{n+4}{2} = 91$, for $n \in \mathbf{N}$.

4 (b) (i) The nth term of an arithmetic series is $3n + 2$.
Find S_n, the sum of the first n terms, in terms of n.

(ii) Evaluate, in terms of n, $\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$.

4 (c) Let $f(x) = \sum_{n=1}^{\infty} q^{n-1}x^n$, where $|x| < 1$ and $0 < q < 1$.

Show that $f(x) = \frac{x}{1-qx}$.

If $g(x) = \frac{1}{1-(1-q)f(x)}$, show that $g(x) = \frac{1-qx}{1-x}$.

SOLUTION

4 (a)

$\binom{n+4}{2} = 91 \Rightarrow \frac{(n+4)(n+3)}{2\times 1} = 91$

$\Rightarrow (n+4)(n+3) = 182$

$\Rightarrow n^2 + 7n + 12 = 182$

$\Rightarrow n^2 + 7n - 170 = 0$

$\Rightarrow (n-10)(n+17) = 0$

$\therefore n = 10$

4 (b) (i)

$u_n = 3n + 2$

$\Rightarrow u_1 = a = 3(1) + 2 = 5$

$\Rightarrow u_2 = 3(2) + 2 = 8$

$\therefore d = u_2 - u_1 = 8 - 5 = 3$

$S_n = \frac{n}{2}[2(5) + (n-1)3]$ $\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$ 3

$\Rightarrow S_n = \frac{n}{2}[10 + 3n - 3]$

$\therefore S_n = \frac{n}{2}(3n + 7)$

4 (b) (i)

> **STEPS**
> 1. Write the general term as a difference of two terms.
> 2. Build a sum table.

1. $S_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ [Already written as a difference of 2 terms.]

2. $u_1 = \frac{1}{1} - \cancel{\frac{1}{2}}$

$u_2 = \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

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$u_{n-1} = \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}$

$u_n = \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$\therefore S_n = 1 - \frac{1}{n+1}$

4 (c)

$$f(x) = \sum_{n=1}^{\infty} q^{n-1}x^n$$

$$\Rightarrow f(x) = q^0x^1 + q^1x^2 + q^2x^3 + q^3x^4 + q^4x^5 + \ldots\ldots$$

$$\Rightarrow f(x) \quad = x + qx^2 + q^2x^3 + q^3x^4 + q^4x^5 + \ldots\ldots$$

$$\Rightarrow qx \times f(x) = \quad qx^2 + q^2x^3 + q^3x^4 + q^4x^5 + \ldots\ldots.$$

$$\therefore f(x) - qx \times f(x) = x$$

$$\Rightarrow (1-qx)f(x) = x$$

$$\therefore f(x) = \frac{x}{1-qx}$$

$$g(x) = \frac{1}{1-(1-q)f(x)}$$

$$\Rightarrow g(x) = \frac{1}{1-(1-q)\left(\frac{x}{1-qx}\right)} \times \frac{(1-qx)}{(1-qx)}$$

$$\Rightarrow g(x) = \frac{1-qx}{(1-qx) - x(1-q)}$$

$$\Rightarrow g(x) = \frac{1-qx}{1 - \cancel{qx} - x + \cancel{xq}}$$

$$\therefore g(x) = \frac{1-qx}{1-x}$$

5 (a) Find the coefficient of a^3 in $(2+a)^5$.

5 (b) (i) Solve the equation $\sqrt{2x+7} = 2+\sqrt{x}$.

(ii) If $x > 0$ and $x \neq 1$, show that

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\log_{30} x}.$$

Note: $\log_b a = \dfrac{\log_c a}{\log_c b}$.

5 (c) Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.

Solution

5 (a)

$$\boxed{u_{r+1} = {}^nC_r (x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r} \text{ .... } 10$$

General term: $u_{r+1} = \binom{5}{r}(2)^{5-r} a^r$

Coefficient of a^3: $r = 3$

$\therefore u_4 = \binom{5}{3}(2)^{5-3} a^3 = 10 \times 2^2 \times a^3 = 40a^3$

Therefore, coefficient of a^3 is 40.

5 (b) (i)

$\sqrt{2x+7} = 2+\sqrt{x}$ [Square both sides.]

$\Rightarrow (\sqrt{2x+7})^2 = (2+\sqrt{x})^2$

$\Rightarrow 2x+7 = 4+4\sqrt{x}+x$ [Isolate the surd.]

$\Rightarrow x+3 = 4\sqrt{x}$ [Square both sides.]

$\Rightarrow (x+3)^2 = (4\sqrt{x})^2$

$\Rightarrow x^2+6x+9 = 16x$

$\Rightarrow x^2-10x+9 = 0$

$\Rightarrow (x-1)(x-9) = 0$

$\therefore x = 1, 9$ [Check both answers.]

Check $x = 1$.	Check $x = 9$.
$\sqrt{2(1)+7} = 2+\sqrt{1}$	$\sqrt{2(9)+7} = 2+\sqrt{9}$
$\sqrt{9} = 2+1$	$\sqrt{25} = 2+3$
$3 = 3$	$5 = 5$

Both answers are correct.

5 (b) (ii)

$$\frac{1}{\log_2 x}+\frac{1}{\log_3 x}+\frac{1}{\log_5 x}$$
$$=\log_x 2+\log_x 3+\log_x 5$$
$$=\log_x (2\times 3\times 5)$$
$$=\log_x 30$$
$$=\frac{1}{\log_{30} x}$$

Log Rules

1. $\log_a M+\log_a N=\log_a (MN)$

4. $\log_a M=\dfrac{\log_b M}{\log_b a}$ [Used to change base]

6. $\log_a b=\dfrac{\log_b b}{\log_b a}=\dfrac{1}{\log_b a}$

5 (c)

Steps

1. Prove result is true for some starting value of $n\in \mathbf{N}_0$.
2. Assuming result is true for $n = k$.
3. Prove result is true for $n = (k+1)$.

1. Prove this statement is true for $n = 1$.

$$\sum_{r=1}^{1} r^2=1^2=1$$

$$\tfrac{n}{6}(n+1)(2n+1)=\tfrac{1}{6}(1+1)(2(1)+1)=\tfrac{1}{6}(2)(3)=1$$

2. Assume it is true for $n = k$.

$$\sum_{r=1}^{k} r^2=\underline{\{1^2+2^2+\ldots+k^2\}=\tfrac{k}{6}(k+1)(2k+1)}$$

3. Prove for $n = k + 1$.

$$\sum_{r=1}^{k+1} (r+1)^2=\underline{\{1^2+2^2+\ldots+k^2\}}+(k+1)^2=\tfrac{k+1}{6}(k+2)(2k+3)$$

Use the result in Step 2 to prove step 3.

$$\Rightarrow \tfrac{k}{6}(k+1)(2k+1)+(k+1)^2=\tfrac{k+1}{6}(k+2)(2k+3)$$

$$\Rightarrow \tfrac{k}{6}(k+1)(2k+1)+(k+1)^2$$
$$=(k+1)[\tfrac{k}{6}(2k+1)+(k+1)]$$
$$=(k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]$$
$$=(k+1)\left[\frac{2k^2+k+6k+6}{6}\right]=(k+1)\left[\frac{2k^2+7k+6}{6}\right]$$
$$=(k+1)\left[\frac{(2k+3)(k+2)}{6}\right]$$
$$=\tfrac{1}{6}(k+1)(2k+3)(k+2)$$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n\in \mathbf{N}_0$.