1998

4 (a) Find the sum to infinity of the geometric series

$1+(\frac{2}{3})+(\frac{2}{3})^2+(\frac{2}{3})^3+\ldots$

(b) If for all integers n,

$u_n = 3+n(n-1)^2$,

show that

$u_{n+1}-u_n = 3n^2-n.$

(c) Show that for n a natural number $\dfrac{1}{4n^2-1}=\dfrac{1}{2}\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)$

Let $u_n = \dfrac{1}{4n^2-1}$.

Find $\sum_{n=1}^{\infty} u_n$.

Find the least value of r such that

$$\sum_{n=1}^{r} u_n > \frac{99}{100}\sum_{n=1}^{\infty} u_n, r \in \mathbf{N}.$$

SOLUTION

4 (a)

$a=1,\ r=\frac{2}{3}$

$S_\infty = \dfrac{a}{1-r},\ -1<r<1$ 6

$$S_\infty = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$$

4 (b)

$u_n = 3+n(n-1)^2$

$\therefore u_{n+1} = 3+(n+1)(n+1-1)^2 = 3+(n+1)n^2$

$u_{n+1}-u_n =$

$= 3+(n+1)n^2-[3+n(n-1)^2]$

$= \cancel{3}+(n+1)n^2-\cancel{3}-n(n-1)^2$

$= n^3+n^2-n(n^2-2n+1)$

$= \cancel{n^3}+n^2-\cancel{n^3}+2n^2-n$

$= 3n^2-n$

4 (c)

$$\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

$$=\frac{1}{2}\left(\frac{1(2n+1)-1(2n-1)}{(2n-1)(2n+1)}\right)$$

$$=\frac{1}{2}\left(\frac{\cancel{2n}+1-\cancel{2n}+1}{(2n-1)(2n+1)}\right)$$

$$=\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{(2n-1)(2n+1)}\right)=\frac{1}{(2n-1)(2n+1)}$$

$$=\frac{1}{4n^2-1}$$

$$S_n=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

$$u_1=\frac{1}{2}\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)$$

$$u_2=\frac{1}{2}\left(\cancel{\frac{1}{3}}-\frac{1}{5}\right)$$

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$$u_{n-1}=\frac{1}{2}\left(\frac{1}{2n-3}-\cancel{\frac{1}{2n-1}}\right)$$

$$u_n=\frac{1}{2}\left(\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}\right)$$

$$\therefore S_n=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)\Rightarrow S_\infty=\frac{1}{2}$$

$$\sum_{n=1}^{r}u_n>\frac{99}{100}\sum_{n=1}^{\infty}u_n$$

$$\Rightarrow\frac{1}{2}\left(1-\frac{1}{2r+1}\right)>\frac{99}{100}\times\frac{1}{2}$$

$$\Rightarrow 1-\frac{1}{2r+1}>\frac{99}{100}$$

$$\Rightarrow 1-\frac{99}{100}>\frac{1}{2r+1}\Rightarrow\frac{1}{100}>\frac{1}{2r+1}$$

Remember for whole positive numbers a, b: If $a\ge b\Rightarrow\frac{1}{a}\le\frac{1}{b}$

$$\Rightarrow 100<2r+1\Rightarrow 99<2r$$

$$\Rightarrow\tfrac{99}{2}<r\Rightarrow r>\tfrac{99}{2}$$

$$\therefore r\ge 50 \text{ as } r\in\mathbf{N}.$$

5 (a) Find the value of the term which is independent of x in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^9.$$

(b) Solve

$\log_5(x-2) = 1 - \log_5(x-6),\ x \in \mathbf{R}, x > 6.$

(c) Let $u_n = (1+x)^n - 1 - nx$ for $n \in \mathbf{N}_0, x \in \mathbf{R}$ and $x > -1$ and where $u_n = u_n(x)$.

Show that

$u_{n+1} \geq u_n$

(i) when $x = 0$

(ii) when $x > 0$

(iii) when $-1 < x < 0$.

Show that $u_2 \geq 0$.

Hence, or otherwise, deduce that

$(1+x)^n \geq 1 + nx,\ x > -1.$

SOLUTION

5 (a)

$$u_{r+1} = \binom{9}{r}(x^2)^{9-r}\left(\frac{1}{x}\right)^r$$

$u_{r+1} = {}^nC_r\,(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r$ 10

$$\Rightarrow u_{r+1} = \binom{9}{r}\frac{x^{18-2r}}{x^r}$$

$$\Rightarrow u_{r+1} = \binom{9}{r}x^{18-3r}$$

Term independent of x: Power of x is zero.

$\therefore 18 - 3r = 0 \Rightarrow r = 6$

$$\therefore u_7 = \binom{9}{6}x^0 = 84$$

5 (b)

$\log_5(x-2) = 1 - \log_5(x-6)$

$\Rightarrow \log_5(x-2) + \log_5(x-6) = 1$

$\Rightarrow \log_5(x-2)(x-6) = 1$

$\Rightarrow (x-2)(x-6) = 5^1$

$\Rightarrow x^2 - 8x + 12 = 5$

$\Rightarrow x^2 - 8x + 7 = 0$

$\Rightarrow (x-1)(x-7) = 0$

$\therefore x = 1, 7$

LOG RULES

1. $\log_a M + \log_a N = \log_a(MN)$

Only use $x = 7$ as $x = 1$ will give you the log of a negative number which is illegal.

5 (c)

$u_n = (1+x)^n - 1 - nx$

$\Rightarrow u_{n+1} = (1+x)^{n+1} - 1 - (n+1)x = (1+x)^{n+1} - 1 - nx - x$

$u_{n+1} \geq u_n \Rightarrow u_{n+1} - u_n \geq 0$

$\Rightarrow (1+x)^{n+1} - \cancel{1} - \cancel{nx} - x - (1+x)^n + \cancel{1} + \cancel{nx} \geq 0$

$\Rightarrow (1+x)^{n+1} - (1+x)^n - x \geq 0$

$\Rightarrow (1+x)^n[(1+x)-1] - x \geq 0$

$\Rightarrow x(1+x)^n - x \geq 0$

$\Rightarrow x[(1+x)^n - 1] \geq 0$

5 (c) (i)

$x = 0$:

$\Rightarrow (0)[(1+0)^n - 1]$

$= 0[0] \geq 0$ [This is true.]

5 (c) (ii)

$x > 0$:

x is a positive number.

$(1+x)^n$ is a positive number greater than 1.

$\therefore [(1+x)^n - 1]$ is a positive number.

$\therefore x[(1+x)^n - 1] \geq 0$

5 (c) (ii)

$-1 < x < 0$:

x is a negative number.

$(1+x)^n$ is a positive number between 0 and 1.

$\therefore [(1+x)^n - 1]$ is a negative number.

$\therefore x[(1+x)^n - 1] \geq 0$

$u_2 = (1+x)^2 - 1 - 2x$

$\Rightarrow u_2 = 1 + 2x + x^2 - 1 - 2x$

$\therefore u_2 = x^2 \geq 0$

You need to deduce that $(1+x)^n \geq 1 + nx$

$\Rightarrow (1+x)^n - 1 - nx \geq 0$

$\Rightarrow u_n \geq 0$

$u_2 \geq 0$

$\Rightarrow u_3 \geq u_2$ [Because you have already proved that $u_{n+1} \geq u_n$.]

$\Rightarrow u_4 \geq u_3$

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$\Rightarrow u_n \geq u_2 \geq 0$