## SEQUENCES & SERIES (Q 4 & 5, PAPER 1)

## 1997

4 (a) Write down, or find, in terms of $n$, the sum of $n$ terms of the finite arithmetic series

$$1 + 2 + 3 + .... + n.$$

(b) If for all integers $n$,

$u_n = (5n-3)2^n$,

verify that

$u_{n+1} - 2u_n = 5(2^{n+1})$.

(c) Consider the sum to $n$ terms, $S_n$, of the following finite geometric series

$S_n = 1 + (1+x) + (1+x)^2 + (1+x)^3 + ... + (1+x)^{n-1}$

for $x > 0$.

Show that the coefficient of $x^2$ in the above expression for $S_n$ is

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + ......\binom{n-1}{2}.$$

By finding $S_{25}$ in terms of $x$ and by considering the coefficient of $x^2$ in $S_{25}$, find the value of $p$ and the value of $q$ for which

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + ......\binom{24}{2} = \binom{p}{q}, \text{ where } p, q \in \mathbf{N}.$$

**SOLUTION**

**4 (a)**

$a = 1,\ d = 1$ $\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$ ....... 3

$\therefore S_n = \frac{n}{2}[2(1) + (n-1)1]$

$\Rightarrow S_n = \frac{n}{2}[2 + n - 1]$

$\therefore S_n = \dfrac{n(n+1)}{2}$

**4 (b)**

$u_n = (5n-3)2^n$

$\Rightarrow u_{n+1} = (5(n+1)-3)2^{n+1} = (5n+2)2^{n+1}$

$u_{n+1} - 2u_n$

$= (5n+2)2^{n+1} - 2(5n-3)2^n$

$= (5n+2)2^{n+1} - (5n-3)2^{n+1}$

$= 2^{n+1}(\cancel{5n} + 2 - \cancel{5n} + 3)$

$= 2^{n+1}(5)$

**4 (c)**

$$(x+y)^n = \binom{n}{0}(x)^n(y)^0 + \binom{n}{1}(x)^{n-1}(y)^1 + \binom{n}{2}(x)^{n-2}(y)^2 \ldots\ldots$$ ....... 9

$$(1+x)^n = \binom{n}{0}(1)^n(x)^0 + \binom{n}{1}(1)^{n-1}(x)^1 + \binom{n}{2}(1)^{n-2}(x)^2 + \binom{n}{3}(1)^{n-3}(x)^3 \ldots$$

$$\Rightarrow (1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 \ldots$$

$$\therefore (1+x)^2 = 1 + 2x + \binom{2}{2}x^2 + \ldots.$$

$$\therefore (1+x)^3 = 1 + 3x + \binom{3}{2}x^2 + \ldots.$$

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$$\therefore (1+x)^{n-1} = 1 + (n-1)x + \binom{n-1}{2}x^2 + \ldots.$$

$\therefore$ Sum of coefficients of $x^2$ : $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots\ldots\binom{n-1}{2}$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$ ....... 5

$a = 1,\ r = (1+x),\ n = 25$

$$\therefore S_{25} = \frac{1(1-(1+x)^{25})}{1-(1+x)}$$

$$\Rightarrow S_{25} = \frac{1-(1+x)^{25}}{-x} = \frac{(1+x)^{25} - 1}{x}$$

$$\Rightarrow S_{25} = \frac{\cancel{1} + 25x + \binom{25}{2}x^2 + \binom{25}{3}x^3 + \binom{25}{4}x^4 + \ldots\ldots + \binom{25}{25}x^{25} - \cancel{1}}{x}$$

$$\Rightarrow S_{25} = 25 + \binom{25}{2}x + \binom{25}{3}x^2 + \binom{25}{4}x^3 + \ldots\ldots + \binom{25}{25}x^{24}$$

$$\therefore \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots.\binom{24}{2} = \binom{25}{3}$$

$\therefore p = 25,\ q = 3$

5 (a) Solve

$$\log_5 x = 1 + \log_5\left(\frac{3}{2x-1}\right), x \in \mathbf{R}, x > \tfrac{1}{2}.$$

(b) (i) Solve $\dfrac{x+3}{x-4} < -2,\ x \neq 4,\ x \in \mathbf{R}.$

(ii) If $k$ is a positive integer and 720 is the coefficient of $x^3$ in the binomial expansion of $(k+2x)^5$, find the value of $k$.

(c) Prove by induction that 8 is a factor of $3^{2n} - 1$ for $n \in \mathbf{N}_0$.

**SOLUTION**

**5 (a)**

$\log_5 x = 1 + \log_5\left(\dfrac{3}{2x-1}\right)$

$\Rightarrow \log_5 x - \log_5\left(\dfrac{3}{2x-1}\right) = 1$

$\Rightarrow \log_5\left(\dfrac{x(2x-1)}{3}\right) = 1$

$\Rightarrow \dfrac{x(2x-1)}{3} = 5^1$

$\Rightarrow x(2x-1) = 15$

$\Rightarrow 2x^2 - x = 15$

$\Rightarrow 2x^2 - x - 15 = 0$

$\Rightarrow (2x+5)(x-3) = 0$

$\therefore x = \cancel{-\tfrac{5}{2}}, 3$

$x$ = 3 is the only answer as you cannot have the logs of negative numbers.

**LOG RULES**

2. $\log_a M - \log_a N = \log_a\left(\dfrac{M}{N}\right)$

**5 (b) (i)**

**STEPS**
**1.** Multiply both sides by the denominator **squared** unless you are certain that it is positive.
**2.** Get all terms on one side and take out the highest common factor.
**3.** Solve the corresponding equation.
**4.** Do region test on the roots in ascending order on **Test Box**.
**5.** Based on the region test write down the solutions.

**1.** $\dfrac{x+3}{x-4} < -2$

$\Rightarrow (x-4)(x+3) < -2(x-4)^2$

**2.** $\Rightarrow (x-4)(x+3) + 2(x-4)^2 < 0$

$\Rightarrow (x-4)[(x+3) + 2(x-4)] < 0$

$\Rightarrow (x-4)[x+3+2x-8] < 0$

$\Rightarrow (x-4)(3x-5) < 0$

**3.** $(x-4)(3x-5)=0 \Rightarrow x=4, \frac{5}{3}$

**4.** Region Test on $(x-4)(3x-5)<0$ .....**Test Box**

| 1 | $\frac{5}{3}$ | 2 | 4 | 5 |
|---|---|---|---|---|
| $(-3)(-2)<0$ | | $(-2)(1)<0$ | | $(1)(10)<0$ |
| Wrong | | Correct | | Wrong |

**5.** $\therefore \frac{5}{3}<x<4$

**5 (b) (ii)**

$$u_{r+1} = {}^nC_r (x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r$$ ....... **10**

$$u_{r+1} = \binom{5}{r} k^{5-r}(2x)^r = \binom{5}{r} k^{5-r} 2^r x^r$$

Coefficient of $x^3 \Rightarrow r=3$

$$\therefore \binom{5}{3} k^{5-3} 2^3 = 720$$

$$\Rightarrow 10k^2(8)=720$$

$$\Rightarrow 80k^2 = 720$$

$$\Rightarrow k^2 = 9$$

$$\therefore k=3$$

**5 (c)**

**Steps**
**1.** Prove result is true for some starting value of $n \in \mathbf{N}_0$.
**2.** Assuming result is true for $n = k$.
**3.** Prove result is true for $n = (k+1)$.

**1.** Prove this statement is true for $n = 1$.

$3^{2(1)}-1=9-1=8$

Therefore, it is true for $n = 1$.

**2.** Assume it is true for $n = k$.

$\Rightarrow 3^{2k}-1=8m,\ m \in \mathbf{N}_0$ (i.e. multiple of 8)

$\Rightarrow 3^{2k}=8m+1$

**3.** Prove for $n = k + 1$.

$3^{2(k+1)}-1=3^{2k+2}-1$

$=3^2 \times 3^{2k}-1$ [Use step 2.]

$=9(8m+1)-1$

$=72m+9-1=72m+8$

$=8(9m+1)$

$=8a,\ a \in \mathbf{N}_0$