1996

4 (a) Find S_n, the sum of n terms, of the geometric series

$$2+\frac{2}{3}+\frac{2}{3^2}+\ldots\ldots\ldots\ldots+\frac{2}{3^{n-1}}.$$

If $S_n=\frac{242}{81}$, find the value of n.

(b) (i) Show that $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is equal to $\sqrt{n+1}-\sqrt{n}$.

(ii) If $u_n=\frac{1}{\sqrt{n+1}+\sqrt{n}}$ find an expression for the sum of the first n terms in terms of n.

(c) $u_1, u_2, u_3, \ldots\ldots\ldots u_n$ is a sequence, where $u_n=1+2+3+\ldots\ldots+n$.

(i) Show $u_n=\frac{n}{2}(n+1)$.

(ii) Express u_n-u_{n-1} in terms of n.

(iii) Show $u_n+u_{n-1}=n^2$.

(iv) Find $u_n^{\ 2}-u_{n-1}^{\ 2}$.

Hence, show that $\sum_{1}^{n}(u_n^{\ 2}-u_{n-1}^{\ 2})=1+2^3+3^3+\ldots.+n^3$ where $u_0=0$.

SOLUTION

4 (a)

$$a=2,\ r=\frac{\frac{2}{3}}{2}=\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$$

$$S_n=\frac{2(1-(\frac{1}{3})^n)}{1-\frac{1}{3}}$$

$$\Rightarrow S_n=\frac{2(1-(\frac{1}{3})^n)}{\frac{2}{3}}$$

$$\therefore S_n=3(1-(\tfrac{1}{3})^n)$$

$$S_n = \frac{242}{81} \Rightarrow 3(1-(\tfrac{1}{3})^n) = \frac{242}{81}$$

$$\Rightarrow (1-(\tfrac{1}{3})^n) = \frac{242}{243}$$

$$\Rightarrow 1-\frac{242}{243} = (\tfrac{1}{3})^n$$

$$\Rightarrow \frac{1}{243} = \left(\frac{1}{3}\right)^n$$

$$\Rightarrow \left(\frac{1}{3}\right)^5 = \left(\frac{1}{3}\right)^n$$

$$\therefore n = 5$$

4 (b) (i)

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{(\sqrt{n+1}+\sqrt{n})} \times \frac{(\sqrt{n+1}-\sqrt{n})}{(\sqrt{n+1}-\sqrt{n})}$$

$$= \frac{(\sqrt{n+1}-\sqrt{n})}{(\sqrt{n+1})^2-(\sqrt{n})^2}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n}$$

$$= \sqrt{n+1}-\sqrt{n}$$

4 (b) (ii)

$u_1: \quad \sqrt{2} - \sqrt{1}$

$u_2: \quad \sqrt{3} - \sqrt{2}$

$u_3: \quad \sqrt{4} - \sqrt{3}$

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$u_{n-2}: \sqrt{n-1} - \sqrt{n-2}$

$u_{n-1}: \sqrt{n} - \sqrt{n-1}$

$u_n: \quad \sqrt{n+1} - \sqrt{n}$

$$\therefore S_n = \sqrt{n+1} - \sqrt{1} = \sqrt{n+1} - 1$$

4 (c) (i)

$a = 1,\ d = 1$

$u_n = S_n = \frac{n}{2}[2(1) + (n-1)1]$

$\Rightarrow u_n = \frac{n}{2}[2 + n - 1]$

$\therefore u_n = \frac{n}{2}(n+1)$

4 (c) (ii)

$u_n = \frac{n}{2}(n+1) \Rightarrow u_{n-1} = \frac{n-1}{2}(n)$

$\therefore u_n - u_{n-1} = \frac{n}{2}(n+1) - \frac{n-1}{2}(n)$

$\Rightarrow u_n - u_{n-1} = \frac{n}{2}[(n+1) - (n-1)]$

$\Rightarrow u_n - u_{n-1} = \frac{n}{2}[n + 1 - n + 1]$

$\Rightarrow u_n - u_{n-1} = \frac{n}{2}[2]$

$\therefore u_n - u_{n-1} = n$

4 (c) (iii)

$u_n + u_{n-1} = \frac{n}{2}(n+1) + \frac{n-1}{2}(n)$

$\Rightarrow u_n + u_{n-1} = \frac{n}{2}[(n+1) + (n-1)]$

$\Rightarrow u_n + u_{n-1} = \frac{n}{2}[n + 1 + n - 1]$

$\Rightarrow u_n + u_{n-1} = \frac{n}{2}[2n]$

$\therefore u_n + u_{n-1} = n^2$

4 (c) (iv)

$\therefore (u_n)^2 - (u_{n-1})^2 = [\frac{n}{2}(n+1)]^2 - [\frac{n-1}{2}(n)]^2$

$\Rightarrow (u_n)^2 - (u_{n-1})^2 = (\frac{n}{2})^2(n+1)^2 - (\frac{n}{2})^2(n-1)^2$

$\Rightarrow (u_n)^2 - (u_{n-1})^2 = (\frac{n}{2})^2[(n+1)^2 - (n-1)^2]$

$\Rightarrow (u_n)^2 - (u_{n-1})^2 = (\frac{n}{2})^2[n^2 + 2n + 1 - n^2 + 2n - 1]$

$$\Rightarrow (u_n)^2 - (u_{n-1})^2 = \left(\frac{n^2}{4}\right)[4n]$$

$\therefore (u_n)^2 - (u_{n-1})^2 = n^3$

$$\sum_{1}^{n} (u_n)^2 - (u_{n-1})^2 = \sum_{1}^{n} n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$$

$$\therefore \sum_{1}^{n} (u_n)^2 - (u_{n-1})^2 = 1 + 2^3 + 3^3 + \ldots\ldots + n^3$$

5 (a) Solve the simultaneous equations

$\log(x+y) = 2\log x$

$\log y = \log 2 + \log(x-1)$ where $x > 1, y > 0$.

(b) (i) Write the binomial expansion of $(a+b)^4$ in ascending powers of b.

Find $\left(x+\frac{1}{x}\right)^4 - \left(x-\frac{1}{x}\right)^4$ in its simplest form.

(ii) Write u_{r+1}, the general term of the binomial expansion of $(3+2x)^n$ in terms of x, r and n.

If the coefficients of x^5 and x^6 are equal, find the v (a)alue of n.

(c) Prove by induction that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}, n \in \mathbf{N}_o$.

SOLUTION

5 (a)

$\log(x+y) = 2\log x$

$\Rightarrow \log(x+y) = \log x^2$

$\therefore x+y = x^2$.....(**1**)

$\log y = \log 2 + \log(x-1)$

$\Rightarrow \log y - \log(x-1) = \log 2$

$\Rightarrow \log\left(\frac{y}{x-1}\right) = \log 2$

$\therefore \frac{y}{x-1} = 2 \Rightarrow y = 2x-2$....(**2**)

Substitute the value of y in Eqn (**2**) into Eqn. (**1**).

$\therefore x + (2x-2) = x^2$

$\Rightarrow 3x - 2 = x^2$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\therefore x = \not{1}, 2$ [You are told that $x > 1$.]

$x = 2 : y = 2(2) - 2 = 2$

5 (b) (i)

$$(x+y)^n = \binom{n}{0}(x)^n(y)^0 + \binom{n}{1}(x)^{n-1}(y)^1 + \binom{n}{2}(x)^{n-2}(y)^2 .... \quad .... \; 9$$

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

$$\Rightarrow (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\left(x+\frac{1}{x}\right)^4 - \left(x-\frac{1}{x}\right)^4$$

$$= x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 - \left[x^4 + 4x^3\left(-\frac{1}{x}\right) + 6x^2\left(-\frac{1}{x}\right)^2 + 4x\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4\right]$$

$$= x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x^2}\right) + 4x\left(\frac{1}{x^3}\right) + \left(\frac{1}{x^4}\right) - \left[x^4 - 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x^2}\right) - 4x\left(\frac{1}{x^3}\right) + \left(\frac{1}{x^4}\right)\right]$$

$$= x^4 + 4x^2 + 6 + \left(\frac{4}{x^2}\right) + \left(\frac{1}{x^4}\right) - x^4 + 4x^2 - 6 + \left(\frac{4}{x^2}\right) - \left(\frac{1}{x^4}\right)$$

$$= 8x^2 + \frac{8}{x^2}$$

5 (b) (ii)

$$u_{r+1} = {}^nC_r(x)^{n-r}(y)^r = \binom{n}{r}(x)^{n-r}(y)^r \quad .... \; 10$$

$$u_{r+1} = \binom{n}{r}(3)^{n-r}(2x)^r \Rightarrow u_{r+1} = \binom{n}{r}3^{n-r}2^r x^r$$

Coefficient of x^5: $r = 5$

$$\therefore u_6 = \binom{n}{5}3^{n-5}2^5x^5 = 32\binom{n}{5}3^{n-5}x^5$$

Coefficient of x^6: $r = 6$

$$\therefore u_7 = \binom{n}{6}3^{n-6}2^6x^6 = 64\binom{n}{6}3^{n-6}x^6$$

$$\therefore 32\binom{n}{5}3^{n-5} = 64\binom{n}{6}3^{n-6}$$

$$\Rightarrow \binom{n}{5} = \frac{64}{32}\times\frac{3^{n-6}}{3^{n-5}}\binom{n}{6}$$

$$\Rightarrow \binom{n}{5} = 2\times3^{-1}\binom{n}{6} \Rightarrow \binom{n}{5} = \frac{2}{3}\binom{n}{6}$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{5\times4\times3\times2\times1} = \frac{2}{3}\times\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6\times5\times4\times3\times2\times1}$$

$$\Rightarrow 1 = \frac{2}{3}\times\frac{n-5}{6} \Rightarrow 1 = \frac{n-5}{9} \Rightarrow 9 = n-5$$

$$\therefore n = 14$$

5 (c)

STEP 1. For n = 1: Prove $y = x^1 \Rightarrow \frac{dy}{dx} = 1$

$$y + \Delta y = x + \Delta x$$
$$y = x$$

$\Delta y = \Delta x$ by subtraction

$$\therefore \frac{\Delta y}{\Delta x} = 1 \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1$$

STEP 2. $n = k$: Assume $y = x^k \Rightarrow \frac{dy}{dx} = kx^{k-1}$

STEP 3. $n = k + 1$: Prove $y = x^{k+1} \Rightarrow \frac{dy}{dx} = (k+1)x^k$

PROOF: $y = x^{k+1} = x^k \times x \Rightarrow \frac{dy}{dx} = x^k \times 1 + x \times kx^{k-1}$ (Product Rule)

$= x^k + kx^k = x^k(k+1)$

Therefore, it is true for $n = k \Rightarrow$ true for $n = k + 1$.

So true for $n = 1$ and true for $n = k \Rightarrow$ true for $n = k + 1 \Rightarrow$ true for all $n \in \mathbf{N}_0$.