Discrete Maths (Q 6 & 7, Paper 2)

2011

6. **(a)** Two adults and four children stand in a row for a photograph. How many different arrangements are possible if the four children are between the two adults?

(b) **(i)** Solve the difference equation $u_{n+2} - 6u_{n+1} + 8u_n = 0$, where $n \geq 0$, given that $u_0 = 0$ and $u_1 = 4$.

(ii) For what value of n is $u_n = 30(2^n)$?

(c) Five cards are drawn together at random from a standard pack of 52 playing cards. Find, in decimal form, correct to two significant figures, the probability that:

(i) all five cards are diamonds
(ii) all five cards are of the same suit
(iii) the five cards are the ace, two, three, four and five of diamonds
(iv) the five cards include the four aces.

Solution

6 (a)

There are 2 ways to arrange the adults at each end and 4 ways to arrange the children in between.
No. of arrangements = $= 2 \times 4! = 2 \times 24 = 48$

> The number of arrangements of n different objects all taken, no repeats = $n!$

6 (b) (i)

> **Steps**
> 1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
> 2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
> 3. Solve this equation to find α, β.
> 4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
> 5. Find l, m using extra conditions (boundary conditions).

$u_n = l(\alpha)^n + m(\beta)^n$

1. $u_{n+2} - 6u_{n+1} + 8u_n = 0$

2. $x^2 - 6x + 8 = 0$

3. $(x-2)(x-4) = 0 \Rightarrow \alpha = 2,\ \beta = 4$

4. $u_n = l(2)^n + m(4)^n$

5. $u_0 = 0 \Rightarrow l + m = 0$....**(1)**

$u_1 = 4 \Rightarrow 2l + 4m = 4 \Rightarrow l + 2m = 2$...**(2)**

Solving Eqns. (**1**) and (**2**) simultaneously $\Rightarrow l = -2,\ m = 2$

$\therefore u_n = -2(2)^n + 2(4)^n = 2(4)^n - 2(2)^n$

6 (b) (ii)

$u_n = 30(2^n)$

$\therefore 2(4)^n - 2(2)^n = 30(2)^n$

$2(2^2)^n - 32(2)^n = 0$

$2(2)^{2n} - 32(2)^n = 0$

$2(2)^n[2^n - 16] = 0$

$[2^n - 16] = 0 \Rightarrow 2^n = 16$

$2^n = 2^4$

$\therefore n = 4$

6 (c) (i)

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)}$$

There are 13 diamonds in a pack of 52 cards.
On the first pick there are 13 diamonds to pick out of 52 cards.
Once a diamond is picked, there are 12 diamonds to pick from 51 cards on the second pick.
And so on.

$p(\text{Diamond, Diamond, Diamond, Diamond, Diamond}) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = 0.00050$

6 (c) (ii)

p(5 diamonds or 5 hearts or 5 clubs or 5 spades)
= 0.0005 + 0.0005 + 0.0005 + 0.0005 = 0.002

6 (c) (iii)

Work out the probability of picking the 5 cards in a particular order and then multiply your answer by the number of ways of arranging the 5 cards as they could have been picked in any order.

p(Ace of diamonds, two of diamonds, three of diamonds, four of diamonds, five of diamonds)

$= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} \times \frac{1}{48} \times 5! = 0.00000038$

6 (c) (iv)

Work out the probability of picking the 5 cards in a particular order and then multiply your answer by the number of ways of arranging the 5 cards as they could have been picked in any order.

p(Ace of diamonds, Ace of hearts, Ace of clubs, Ace of spades, Any card)

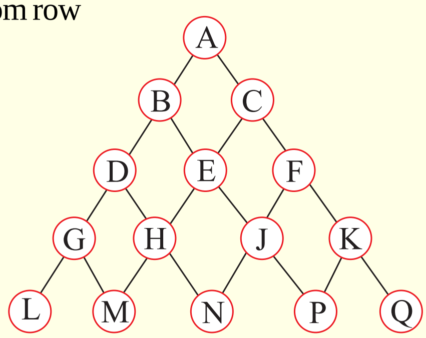
$= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} \times \frac{48}{48} \times 5! = 0.000018$

7. **(a)** A team of four is selected from a group of seven girls and five boys.

(i) How many different selections are possible?

(ii) How many of these selections include at least one girl?

(b) A marble falls down from A and must follow one of the paths indicated on the diagram. All paths from A to the bottom row are equally likely to be followed.



(i) One of the paths from A to H is A-B-D-H. List the other two possible paths from A to H.

(ii) Find the probability that the marble passes through H or J.

(iii) Find the probability that the marble lands at N.

(iv) Two marbles fall from A, one after the other, without affecting each other. Find the probability that they both land at P.

(c) The real numbers a, b and c have mean μ and standard deviation σ.

(i) Show that the mean of the numbers $\frac{a-\mu}{\sigma}$, $\frac{b-\mu}{\sigma}$ and $\frac{c-\mu}{\sigma}$ is 0.

(ii) Find, with justification, the standard deviation of the numbers $\frac{a-\mu}{\sigma}$, $\frac{b-\mu}{\sigma}$ and $\frac{c-\mu}{\sigma}$.

SOLUTION

7 (a) (i)

Number of teams = ${}^{12}C_4 = 495$

7G 5B
↓
4 Places

The number of selections of n different objects taking r at a time $= {}^nC_r$

7 (a) (ii)

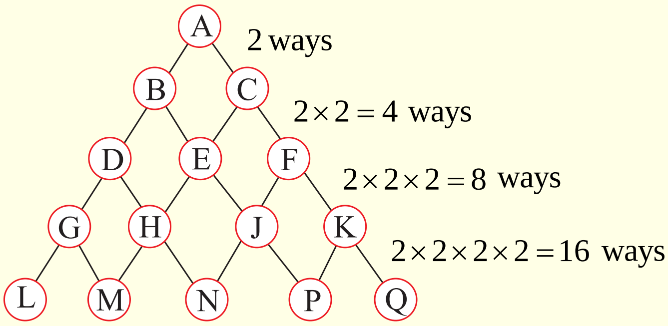
How many teams can you pick with boys only?

No. of boys only teams = ${}^7C_0 \times {}^5C_4 = 5$

7G	5B
↓	↓
0 Places	4 Places

Therefore, the number of teams with at least one girl = 495 – 5 = 490

7 (b) There are 16 ways for the marble to start at A and reach the bottom.



7 (b) (i)
A-B-E-H, A-C-E-H

7 (b) (ii)

A-B-D-**H**-M, A-B-E-**H**-M, A-C-E-**H**-M
A-B-D-**H**-N, A-B-E-**H**-V, A-C-E-**H**-N

There are 6 ways the marble passes through **H**.

A-B-E-**J**-N, A-C-E-**J**-N, A-C-F-**J**-N
A-B-E-**J**-P, A-C-E-**J**-P, A-C-F-**J**-P

There are 6 ways the marble passes through **J**.

p(Marble passes through H or J) $= \frac{12}{16} = \frac{3}{4}$

7 (b) (iii)

A-B-D-H-**N**, A-B-E-H-**N**, A-C-E-H-**N**
A-B-E-J-**N**, A-C-E-J-**N**, A-C-F-J-**N**

There are 6 ways the marble lands on **N**.

p(Marble lands at N) $= \frac{6}{16} = \frac{3}{8}$

7 (b) (iv)

A-C-F-K-**P**, A-C-F-J-**P**,
A-C-E-J-**P**, A-B-E-J-**P**

There are 4 ways the marble lands on **P**.

There are 16 ways in total for the marble to start at A and fall to the bottom.

p(A marble landing at P and then another marble landing at P) $= \frac{4}{16} \times \frac{4}{16} = \frac{1}{16}$

7 (c)

Numbers a, b, c

Mean $\mu = \dfrac{a+b+c}{3} \Rightarrow (a+b+c) = 3\mu$

$$\overline{x} = \frac{x_1 + x_2 + \dots\dots\dots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$

x	d	d^2
a	$(a-\mu)$	$(a-\mu)^2$
b	$(b-\mu)$	$(b-\mu)^2$
c	$(c-\mu)$	$(c-\mu)^2$

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$

where deviation $d = (x - \overline{x}) = (\text{Number} - \text{Mean})$

Standard Deviation $\sigma = \sqrt{\dfrac{(a-\mu)^2 + (b-\mu)^2 + (c-\mu)^2}{3}}$

7 (c) (i)

Numbers: $\dfrac{a-\mu}{\sigma}, \dfrac{b-\mu}{\sigma}, \dfrac{c-\mu}{\sigma}$

$$\text{Mean} = \frac{\frac{a-\mu}{\sigma} + \frac{b-\mu}{\sigma} + \frac{c-\mu}{\sigma}}{3}$$

$$= \frac{a-\mu+b-\mu+c-\mu}{3\sigma}$$

$$= \frac{(a+b+c)-3\mu}{3\sigma}$$

$$= \frac{3\mu - 3\mu}{3\sigma}$$

$$= 0$$

7 (c) (ii)

Numbers: $\dfrac{a-\mu}{\sigma}, \dfrac{b-\mu}{\sigma}, \dfrac{c-\mu}{\sigma}$

x	d	d^2
$\frac{a-\mu}{\sigma}$	$\frac{a-\mu}{\sigma} - 0$	$\frac{(a-\mu)^2}{\sigma^2}$
$\frac{b-\mu}{\sigma}$	$\frac{b-\mu}{\sigma} - 0$	$\frac{(b-\mu)^2}{\sigma^2}$
$\frac{c-\mu}{\sigma}$	$\frac{c-\mu}{\sigma} - 0$	$\frac{(c-\mu)^2}{\sigma^2}$

$$\text{Standard Deviation} = \sqrt{\frac{\frac{(a-\mu)^2}{\sigma^2} + \frac{(b-\mu)^2}{\sigma^2} + \frac{(c-\mu)^2}{\sigma^2}}{3}}$$

$$= \sqrt{\frac{(a-\mu)^2 + (b-\mu)^2 + (c-\mu)^2}{3\sigma^2}}$$

$$= \frac{\sqrt{\frac{(a-\mu)^2 + (b-\mu)^2 + (c-\mu)^2}{3}}}{\sigma}$$

$$= \frac{\sigma}{\sigma}$$

$$= 1$$