DISCRETE MATHS (Q 6 & 7, PAPER 2)

2009

6 (a) A student taking a literature course has to read three novels from a list of ten novels.

(i) How many different selections of three novels are possible?

(ii) Two of the ten novels are by the same author. How many selections are possible if the student wishes to choose three novels by different authors?

(b) (i) In how many different ways can eight people be seated in a row?

(ii) Three girls and five boys sit in a row, arranged at random. Find the probability that the three girls are seated together.

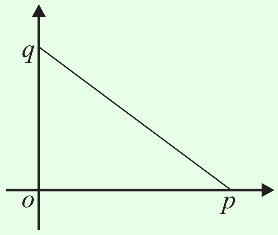
(iii) Three girls and n boys sit in a row, arranged at random. If the probability that the three girls are seated together is $\frac{1}{35}$, find the value of n.

(c) x and y are randomly selected integers with
p is the point with coordinates $(x, 0)$ and q is the point with coordinates $(0, y)$.
Find the probability that

(i) the slope of pq is equal to –1

(ii) the slope of pq is greater than –1

(iii) the length of $[pq]$ is less than or equal to 5.



SOLUTION

6 (a) (i)

How many ways can you pick 3 novels from 10 novels?

$$^{10}C_3 = \frac{10\times 9\times 8}{3\times 2\times 1} = 120$$

The number of selections of n different objects taking r at a time $= {}^nC_r$

6 (a) (ii)

How many ways from picking your 3 novels can you pick the 2 novels written by the same author? Put aside the 2 novels that have already been picked. So your choice is now how many ways you can pick the remaining book from 8 novels.

$$^{8}C_1 = 8$$

Therefore, the number of ways of picking 3 novels from 10 novels with different authors:

$$^{10}C_3 - {}^{8}C_1 = 120 - 8 = 112$$

6 (b) (i)

No. of ways of arranging 8 people = 8! = 40,320

The number of arrangements of n different objects all taken, no repeats = $n!$

6 (b) (ii)

Glue the three girls together and treat them as a single team. The number of ways of arranging the 5 boys and team of girls is 6!
The number of ways of arranging the three girls in the team is 3!
The number of ways of arranging 5 boys and 3 girls with the 3 girls seated together
$= 6! \times 3! = 4320$

$$p(\text{3 girls seated together}) = \frac{4320}{40320} = \frac{3}{28}$$

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)}$$

6 (b) (iii)

The number of ways of arranging n boys and team of girls is $(n + 1)!$
The number of ways of arranging the three girls in the team is 3!
The number of ways of arranging n boys and 3 girls with the 3 girls seated together
$= 3! \times (n+1)!$

$$p(\text{3 girls seated together}) = \frac{3! \times (n+1)!}{(n+3)!} = \frac{1}{35}$$

$$\therefore \frac{6}{(n+3)(n+2)} = \frac{1}{35}$$

$$210 = (n+3)(n+2)$$

$$210 = n^2 + 5n + 6$$

$$0 = n^2 + 5n - 204$$

$$0 = (n+17)(n-12)$$

$$\therefore n = \cancel{-17}, 12$$

6 (c) (i)

Points: $p(x, 0)$, $q(0, y)$

$$m = \frac{y-0}{0-x} = -\frac{y}{x}$$

The slope is –1 if $x = y$. Draw up a table of all the possibilities.
There are 100 possible outcomes.
Highlight the points where $x = y$.
There are 10 points where $x = y$.

$$p(\text{Slope} = -1) = \frac{10}{100} = \frac{1}{10}$$

y values

x values	1	2	3	4	5	6	7	8	9	10
1	**(1, 1)**	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	(1, 10)
2	(2, 1)	**(2, 2)**	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	(2, 10)
3	(3, 1)	(3, 2)	**(3, 3)**	(3, 4)	(3, 5)	(3, 6)	(3, 7)	(3, 8)	(3, 9)	(3, 10)
4	(4, 1)	(4, 2)	(4, 3)	**(4, 4)**	(4, 5)	(4, 6)	(4, 7)	(4, 8)	(4, 9)	(4, 10)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	**(5, 5)**	(5, 6)	(5, 7)	(5, 8)	(5, 9)	(5, 10)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	**(6, 6)**	(6, 7)	(6, 8)	(6, 9)	(6, 10)
7	(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)	**(7, 7)**	(7, 8)	(7, 9)	(7, 10)
8	(8, 1)	(8, 2)	(8, 3)	(8, 4)	(8, 5)	(8, 6)	(8, 7)	**(8, 8)**	(8, 9)	(8, 10)
9	(9, 1)	(9, 2)	(9, 3)	(9, 4)	(9, 5)	(9, 6)	(9, 7)	(9, 8)	**(9, 9)**	(9, 10)
10	(10, 1)	(10, 2)	(10, 3)	(10, 4)	(10, 5)	(10, 6)	(10, 7)	(10, 8)	(10, 9)	**(10, 10)**

6 (c) (ii)

$$-\frac{y}{x} > -1 \Rightarrow \frac{y}{x} < 1 \Rightarrow y < x$$

The slope is greater than –1 for all $y < x$.
Highlight the points where $y < x$.
There are 45 points where $y < x$.

$$p(\text{Slope} > -1) = \frac{45}{100} = \frac{9}{20}$$

y values

x values	1	2	3	4	5	6	7	8	9	10
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	(1, 10)
2	**(2, 1)**	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	(2, 10)
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5	**(5, 1)**	**(5, 2)**	**(5, 3)**	**(5, 4)**	(5, 5)	(5, 6)	(5, 7)	(5, 8)	(5, 9)	(5, 10)
6	**(6, 1)**	**(6, 2)**	**(6, 3)**	**(6, 4)**	**(6, 5)**	(6, 6)	(6, 7)	(6, 8)	(6, 9)	(6, 10)
7	**(7, 1)**	**(7, 2)**	**(7, 3)**	**(7, 4)**	**(7, 5)**	**(7, 6)**	(7, 7)	(7, 8)	(7, 9)	(7, 10)
8	**(8, 1)**	**(8, 2)**	**(8, 3)**	**(8, 4)**	**(8, 5)**	**(8, 6)**	**(8, 7)**	(8, 8)	(8, 9)	(8, 10)
9	**(9, 1)**	**(9, 2)**	**(9, 3)**	**(9, 4)**	**(9, 5)**	**(9, 6)**	**(9, 7)**	**(9, 8)**	(9, 9)	(9, 10)
10	**(10, 1)**	**(10, 2)**	**(10, 3)**	**(10, 4)**	**(10, 5)**	**(10, 6)**	**(10, 7)**	**(10, 8)**	**(10, 9)**	(10, 10)

6 (c) (iii)

$$\sqrt{(0-x)^2+(y-0)^2} \le 5 \Rightarrow \sqrt{x^2+y^2} \le 5 \Rightarrow x^2+y^2 \le 25$$

The distance |pq| is less than or equal to 5 for all values of $x^2+y^2 \le 25$.
Highlight the points where this condition holds.
There are 15 points where this condition holds.

$$p(|pq| \le 25) = \frac{15}{100} = \frac{3}{20}$$

y values

x values

	1	2	3	4	5	6	7	8	9	10
1	**(1, 1)**	**(1, 2)**	**(1, 3)**	**(1, 4)**	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	(1, 10)
2	**(2, 1)**	**(2, 2)**	**(2, 3)**	**(2, 4)**	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	(2, 10)
3	**(3, 1)**	**(3, 2)**	**(3, 3)**	**(3, 4)**	(3, 5)	(3, 6)	(3, 7)	(3, 8)	(3, 9)	(3, 10)
4	**(4, 1)**	**(4, 2)**	**(4, 3)**	(4, 4)	(4, 5)	(4, 6)	(4, 7)	(4, 8)	(4, 9)	(4, 10)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, 7)	(5, 8)	(5, 9)	(5, 10)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)	(6, 7)	(6, 8)	(6, 9)	(6, 10)
7	(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)	(7, 7)	(7, 8)	(7, 9)	(7, 10)
8	(8, 1)	(8, 2)	(8, 3)	(8, 4)	(8, 5)	(8, 6)	(8, 7)	(8, 8)	(8, 9)	(8, 10)
9	(9, 1)	(9, 2)	(9, 3)	(9, 4)	(9, 5)	(9, 6)	(9, 7)	(9, 8)	(9, 9)	(9, 10)
10	(10, 1)	(10, 2)	(10, 3)	(10, 4)	(10, 5)	(10, 6)	(10, 7)	(10, 8)	(10, 9)	(10, 10)

7 (a) The prices of four food items in a shopping basket are €3, €5, €1 and €6.
Find the weighted mean price of these items using the weights 2, 3, 4 and 1 respectively.

(b) (i) Solve the difference equation $u_{n+2} - 6u_{n+1} + 5u_n = 0$, where $n \ge 1$, given that $u_1 = 0$ and $u_2 = 20$.

(ii) Find an expression in *n* for the sum of the terms $u_1 + u_2 + u_3 + ... + u_n$.

(c) The two numbers *a* and *b* have mean $\bar{x}$ and standard deviation σ_1.
The three numbers *c*, *d* and *e* have mean $\bar{x}$ and standard deviation σ_2.
Find the standard deviation of the five numbers *a*, *b*, *c*, *d* and *e* in terms of σ_1 and σ_2.

SOLUTION

7 (a)

x	w	wx
€3	2	6
€5	3	15
€1	4	4
€6	1	6
	10	31

$$\bar{w} = \frac{31}{10} = 3.1$$

ANS: €3.10

$$\bar{w} = \frac{\sum wx}{\sum w}$$

7 (b) (i)

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

$$u_n = l(\alpha)^n + m(\beta)^n$$

1. $u_{n+2} - 6u_{n+1} + 5u_n = 0$

2. $x^2 - 6x + 5 = 0$

3. $(x-1)(x-5) = 0$

$x = 1,\ 5$

$\alpha = 1,\ \beta = 5$

4. $u_n = l(1)^n + m(5)^n = l + m(5)^n$

5. $u_1 = 0 \Rightarrow l + m(5)^1 = 0 \Rightarrow l + 5m = 0 \Rightarrow l = -5m$....(**1**)

$u_2 = 20 \Rightarrow l + m(5)^2 = 20 \Rightarrow l + 25m = 20$....(**2**)

Substitute Eqn. (**1**) into Eqn. (**2**):

$(-5m) + 25m = 20$

$20m = 20$

$\therefore m = 1$

$\therefore l = -5$

$u_n = -5 + (1)(5)^n$

$u_n = 5^n - 5$

7 (b) (ii)

$$u_1 + u_2 + u_3 + \dots\dots\dots\dots + u_n$$
$$= (5^1 - 5) + (5^2 - 5) + (5^3 - 5) + \dots\dots\dots + (5^n - 5)$$
$$= (5^1 + 5^2 + 5^3 + \dots\dots + 5^n) - 5n$$
$$= \tfrac{5}{4}(5^n - 1) - 5n$$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

Consider the expression in brackets.
It is a geometric series.

$a = 5,\ r = 5$

$$S_n = \frac{5(1-5^n)}{1-5} = \frac{5(1-5^n)}{-4}$$
$$= \tfrac{5}{4}(5^n - 1)$$

7 (c)

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$

where deviation $d = (x - \bar{x}) = (\text{Number} - \text{Mean})$

x	d	d^2
a	$(a-\bar{x})$	$(a-\bar{x})^2$
b	$(b-\bar{x})$	$(b-\bar{x})^2$

$$\sigma_1 = \sqrt{\frac{(a-\bar{x})^2 + (b-\bar{x})^2}{2}}$$
$$\sigma_1^2 = \frac{(a-\bar{x})^2 + (b-\bar{x})^2}{2}$$
$$2\sigma_1^2 = (a-\bar{x})^2 + (b-\bar{x})^2$$

x	d	d^2
c	$(c-\bar{x})$	$(c-\bar{x})^2$
d	$(d-\bar{x})$	$(d-\bar{x})^2$
e	$(e-\bar{x})$	$(e-\bar{x})^2$

$$\sigma_2 = \sqrt{\frac{(c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2}{3}}$$
$$\sigma_2^2 = \frac{(c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2}{3}$$
$$3\sigma_2^2 = (c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2$$

$$\sigma = \sqrt{\frac{(a-\bar{x})^2 + (b-\bar{x})^2 + (c-\bar{x})^2 + (d-\bar{x})^2 + (e-\bar{x})^2}{5}}$$
$$= \sqrt{\frac{2\sigma_1^2 + 3\sigma_2^2}{5}}$$