DISCRETE MATHS (Q 6 & 7, PAPER 2)

2008

6 (a) In a certain subject, the examination consists of a project, a practical test, and a written paper. The overall result is the weighted mean of the percentages achieved in these three components, using the weights 2, 3 and 5, respectively. Michael scores 65% in the project and 80% in the practical.
What percentage mark must he get in the written paper in order to get an overall result of 70%?

(b) Solve the difference equation $u_{n+2} - 4u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 1$ and $u_1 = 2$.

(c) A bag contains discs of three different colours.
There are 5 red discs, 1 white disc and x black discs.
Three discs are picked together at random.

(i) Write down an expression in x for the probability that the three discs are all different in colour.

(ii) If the probability that the three discs are all different in colour is equal to the probability that they are all black, find x.

SOLUTION

5 (a)

x	w	wx
65	2	130
80	3	240
y	5	$5y$
	10	$5y + 370$

WEIGHTED MEAN $\bar{w}$

$$\bar{w} = \frac{\sum wx}{\sum w}$$ 5

$$\frac{5y+370}{10} = 70 \Rightarrow 5y + 370 = 700$$
$$\Rightarrow 5y = 330$$
$$\therefore y = 66\%$$

5 (b)

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

$$u_n = l(\alpha)^n + m(\beta)^n$$ 1

1. $u_{n+2} - 4u_{n+1} + u_n = 0$

2. $x^2 - 4x + 1 = 0$

3. $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$

$\therefore x = 2 \pm \sqrt{3} \Rightarrow \alpha = 2 + \sqrt{3},\ \beta = 2 - \sqrt{3}$

4. $\therefore u_n = l(2+\sqrt{3})^n + m(2-\sqrt{3})^n$

5. $u_0 = 1 \Rightarrow l + m = 1 \Rightarrow m = 1 - l$

$u_1 = 2 \Rightarrow l(2+\sqrt{3}) + m(2-\sqrt{3}) = 2$

$\Rightarrow l(2+\sqrt{3}) + (1-l)(2-\sqrt{3}) = 2$

$\Rightarrow 2l + \sqrt{3}l + 2 - \sqrt{3} - 2l + \sqrt{3}l = 2$

$\Rightarrow 2\sqrt{3}l = \sqrt{3}$

$\therefore l = \frac{1}{2} \Rightarrow m = \frac{1}{2}$

$\therefore u_n = \frac{1}{2}(2+\sqrt{3})^n + \frac{1}{2}(2-\sqrt{3})^n$

6 (c) (i)

5 Red, 1 White, x Black

Total: (x + 6) discs

Find the probability of a particular combination and then multiply it by the number of ways in which this combination can take place.

$$p(\text{Red, White, Black}) = \frac{5}{(x+6)} \times \frac{1}{(x+5)} \times \frac{x}{(x+4)} \times 3!$$

$$= \frac{30x}{(x+6)(x+5)(x+4)}$$

6 (c) (ii)

$$p(\text{Black, Black, Black}) = \frac{x}{(x+6)} \times \frac{(x-1)}{(x+5)} \times \frac{(x-2)}{(x+4)}$$

$$\therefore \frac{\cancel{x}(x-1)(x-2)}{\cancel{(x+6)}\,\cancel{(x+5)}\,\cancel{(x+4)}} = \frac{30\cancel{x}}{\cancel{(x+6)}\,\cancel{(x+5)}\,\cancel{(x+4)}}$$

$\Rightarrow (x-1)(x-2) = 30$

$\Rightarrow x^2 - 3x + 2 = 30$

$\Rightarrow x^2 - 3x - 28 = 0$

$\Rightarrow (x-7)(x+4) = 0$

$\therefore x = 7,\ \cancel{-4}$

7 (a) Katie must choose five subjects from nine available subjects.
The nine subjects include French and German.

(i) How many different combinations of five subjects are possible?

(ii) How many different combinations are possible if Katie wishes to study German but not French?

(b) Four cards are drawn together from a pack of 52 playing cards.
Find the probability that

(i) the four cards drawn are the four aces

(ii) two of the cards are clubs and the other two are diamonds

(iii) there are three clubs and two aces among the four cards.

(c) The arithmetic mean of the three numbers x_1, x_2, x_3 is $\bar{x}$.
Let $d_1 = x_1 - \bar{x}$, $d_2 = x_2 - \bar{x}$ and $d_3 = x_3 - \bar{x}$.

(i) Show that $\sum_{r=1}^{3} d_r = 0$.

(ii) The standard deviation of the three numbers x_1, x_2, x_3 is σ.

Given any real number b, let $k^2 = \sum_{r=1}^{3} \frac{(d_r - b)^2}{3}$.

Show that $\sigma^2 = k^2 - b^2$.

SOLUTION

7 (a) (i)

The number of selections of n different objects taking r at a time $= {}^nC_r$ **11**

The number of ways in which you can pick 5 subjects from 9 subjects where order is unimportant $= {}^9C_5 = 126$

7 (a) (ii)

French is excluded and German is already picked as one of her subjects. Therefore, she is picking 4 subjects from 8.

The number of ways in which you can pick 4 subjects from 7 subjects where order is unimportant $= {}^7C_4 = 35$

7 (b)
Find the probability of a particular combination and then multiply it by the number of ways in which this combination can take place.

7 (b) (i)

The probability of picking an ace on the first pick is $\frac{4}{52}$.

The probability of picking an ace on your second pick is $\frac{3}{51}$ as you are down an ace and down a card. Continue like this until the fourth ace is picked.
There is only one arrangement of four aces.

$p(\text{Ace, Ace, Ace, Ace}) = \frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}\times\frac{1}{49} = \frac{1}{270{,}725}$

7 (b) (ii)

The probability of picking a club on your first pick is $\frac{13}{52}$.

The probability of picking a club on your second pick is $\frac{12}{51}$ as you are down one club and down one card.

The probability of picking a diamond on your third pick is $\frac{13}{50}$ as there are 13 diamonds to choose from the 50 cards that are left.

The probability of picking a diamond on your fourth pick is $\frac{12}{49}$ as there are now 12 diamonds to choose from 49 cards.
However, you may not have picked them in this order so you need to multiply your answer by the number of arrangements of 4 objects where 2 are alike and 2 more are alike.

> Number of arrangements of n objects, p alike of one kind, q alike of another kind $= \dfrac{n!}{p!q!}$ **10**

$p(\text{Club, Club, Diamond, Diamond}) = \frac{13}{52}\times\frac{12}{51}\times\frac{13}{50}\times\frac{12}{49}\times\frac{4!}{2!2!} = \frac{468}{20825}$

7 (b) (iii)
If there are three clubs and two Aces among the four cards, then one of the cards has to be an Ace of Clubs.

The probability of picking a club that is not an ace on your first pick is $\frac{12}{52}$.

The probability of picking another club that is not an ace on your second pick is $\frac{11}{51}$.

The probability of picking the ace of clubs on your third pick is $\frac{1}{50}$.

The probability of picking an ace other than the ace of clubs is $\frac{3}{49}$.
Multiply your answer by the number of ways in which you can arrange four objects two of which are the same.

p(Club that's not an Ace, Club that's not an Ace, Ace of Clubs, Ace that is not the Ace of Clubs) $= \frac{12}{52}\times\frac{11}{51}\times\frac{1}{50}\times\frac{3}{49}\times\frac{4!}{2!} = \frac{198}{270725}$

7 (c) (i)

$$\bar{x} = \frac{x_1 + x_2 + \dots\dots\dots\dots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ 3

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$$\sum_{r=1}^{3} d_r = d_1 + d_2 + d_3 = x_1 - \bar{x} + x_2 - \bar{x} + x_3 - \bar{x}$$

$$\Rightarrow \sum_{r=1}^{3} d_r = x_1 + x_2 + x_3 - 3\bar{x}$$

$$\Rightarrow \sum_{r=1}^{3} d_r = x_1 + x_2 + x_3 - 3\left(\frac{x_1 + x_2 + x_3}{3}\right) = 0$$

7 (c) (ii)

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$ 6

x	d	d^2
x_1	$x_1 - \bar{x}$	$(x_1 - \bar{x})^2$
x_2	$x_2 - \bar{x}$	$(x_2 - \bar{x})^2$
x_3	$x_3 - \bar{x}$	$(x_2 - \bar{x})^2$

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3}}$$

$$\Rightarrow \sigma^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3}$$

$$k^2 = \sum_{r=1}^{3} \frac{(d_r - b)^2}{3} = \frac{(d_1 - b)^2}{3} + \frac{(d_2 - b)^2}{3} + \frac{(d_3 - b)^2}{3}$$

$$\Rightarrow k^2 = \frac{((x_1 - \bar{x}) - b)^2 + ((x_2 - \bar{x}) - b)^2 + ((x_3 - \bar{x}) - b)^2}{3}$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 - 2b(x_1 - \bar{x}) + b^2 + (x_2 - \bar{x})^2 - 2b(x_2 - \bar{x}) + b^2 + (x_3 - \bar{x})^2 - 2b(x_3 - \bar{x}) + b^2}{3}$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 - 2b[x_1 - \bar{x} + x_2 - \bar{x} + x_3 - \bar{x}] + 3b^2}{3}$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3} - 2b\frac{[x_1 + x_2 + x_3 - 3\bar{x}]}{3} + b^2$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3} - 2b\left[\frac{x_1 + x_2 + x_3}{3} - \bar{x}\right] + b^2$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3} - 2b[0] + b^2$$

$$\Rightarrow k^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3} + b^2$$

$$\therefore \sigma^2 = k^2 - b^2$$