Discrete Maths (Q 6 & 7, Paper 2)

2000

6 (a) A bank gives each of its customers a four digit personal identification number which is formed from the digits 0 to 9 inclusive. Examples are 2475, 0865 and 3422.

(i) How many different personal identification numbers can the bank use?

(ii) If the bank decides not to use personal identification numbers that begin with 0, how many different numbers can it then use?

6 (b) (i) Solve the difference equation $12u_{n+2} - 8u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = \frac{1}{15}$ and $u_1 = \frac{7}{30}$.

(ii) Hence, express u_3 in the form $\frac{p}{q}$ where $p, q \in \mathbf{N}$.

6 (c) Six red discs, numbered from 1 to 6, and four green discs, numbered from 7 to 10, are placed in box A. Ten blue discs, numbered from 1 to 10, are placed in box B. Two discs are drawn from box A and two discs are drawn from box B. The four discs are drawn at random and without replacement.
Find the probability that the discs drawn are

(i) two red discs and two even numbered blue discs

(ii) one red disc, one green disc and two blue discs with all fours discs odd numbered

(iii) one red disc, one green disc and two blue discs with the total on the red and green discs equal to 10 and the total on the blue discs also equal to 10.

Solution

6 (a) (i)

No. of ways = 10 × 10 × 10 × 10 = 10,000

6 (a) (ii)

No. of ways = 9 × 10 × 10 × 10 = 9,000

Cannot be a zero.

6 (b) (i)

$$u_n = l(\alpha)^n + m(\beta)^n \quad \ldots\ldots \text{(1)}$$

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $12u_{n+2} - 8u_{n+1} + u_n = 0$

2. $12x^2 - 8x + 1 = 0$

3. $(6x-1)(2x-1) = 0 \Rightarrow \alpha = \frac{1}{6},\ \beta = \frac{1}{2}$

4. $u_n = l(\frac{1}{6})^n + m(\frac{1}{2})^n$

5. $u_0 = \frac{1}{15} \Rightarrow u_0 = l + m = \frac{1}{15} \Rightarrow 15l + 15m = 1 \ldots\ldots (\mathbf{1})$

$u_1 = \frac{7}{30} \Rightarrow u_1 = \frac{1}{6}l + \frac{1}{2}m = \frac{7}{30} \Rightarrow 5l + 15m = 7 \ldots\ldots (\mathbf{2})$

Subtracting Eqns. (**1**) and (**2**): $10l = -6 \Rightarrow l = -\frac{3}{5}$

Substitute this value of l into Eqn. (**1**): $15(-\frac{3}{5}) + 15m = 1 \Rightarrow -9 + 15m = 1 \Rightarrow 15m = 10$

$$\therefore m = \frac{2}{3}$$

Ans: $u_n = -\frac{3}{5}(\frac{1}{6})^n + \frac{2}{3}(\frac{1}{2})^n = \frac{2}{3}(\frac{1}{2})^n - \frac{3}{5}(\frac{1}{6})^n$

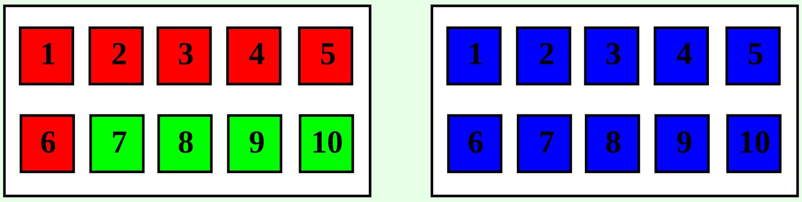
6 (b) (ii)

$u_3 = \frac{2}{3}(\frac{1}{2})^3 - \frac{3}{5}(\frac{1}{6})^3 = \frac{2}{3}(\frac{1}{8}) - \frac{3}{5}(\frac{1}{216})$

$\Rightarrow u_3 = \frac{1}{12} - \frac{1}{360} = \frac{30}{360} - \frac{1}{360}$

$\therefore u_3 = \frac{29}{360}$

6 (c) (i)



The probability of picking a red disc from the first box is $\frac{6}{10}$. There are 6 red discs to pick from a total of 10 discs.

The probability of picking another red disc from the first box is $\frac{5}{9}$. There is one less red disc and one less disc to pick from.

The probability of picking an even blue disc from the second box is $\frac{5}{10}$. There are 5 even blue discs to pick from 10 discs.

The probability of picking a second even blue from the second box is $\frac{4}{9}$.

p(Red, Red) and p(Even, Even) $= \frac{6}{10} \times \frac{5}{9} \times \frac{5}{10} \times \frac{4}{9} = \frac{2}{27}$

6 (c) (ii)

The probability of picking an odd red from the first box is $\frac{3}{10}$. There are 3 odd reds to pick from 10 discs.

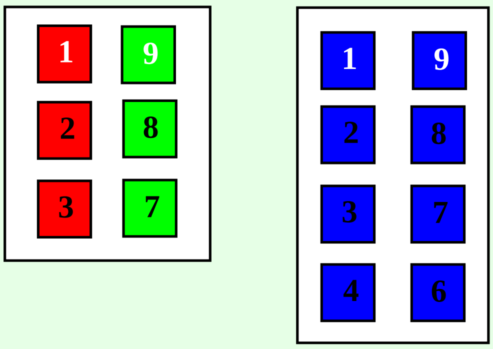
The probability of then picking an odd green from the first box is $\frac{2}{9}$. There are 2 odd green discs to pick from the remaining 9 discs.
However, you could have picked the green first instead of the red. So you need to multiply by 2 to take both options into account.

The probability of picking an odd blue disc from the second box is $\frac{5}{10}$. There are 5 odd blue discs to pick from 10 discs.

The probability of picking a second odd blue from the second box is $\frac{4}{9}$.

p(Odd red, Odd green) and p(Odd blue, Odd blue) $= \frac{3}{10} \times \frac{2}{9} \times 2 \times \frac{5}{10} \times \frac{4}{9} = \frac{4}{135}$

6 (c) (iii)



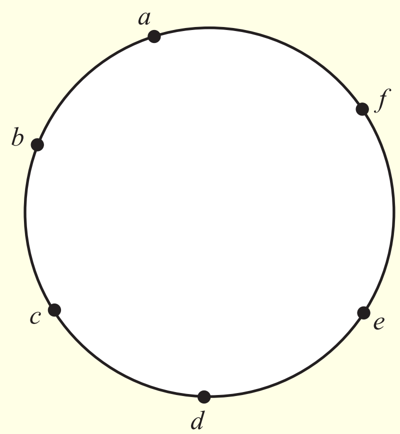
When faced with this situation, write out all the possibilities.
There are 3 possibilities in the first box and 4 possibilities in the second box. Between both boxes there are 4×3 or 12 possibilities.

Work out the probability of picking the 4 discs shown (with white numbers) and multiply your answer by 12.

p(red 1, green 9) and p(blue 1, blue 9) $= \frac{1}{10} \times \frac{1}{9} \times 2 \times \frac{1}{10} \times \frac{1}{9} \times 2 = \frac{1}{2025}$

p(red and green whose sum is 10 from box 1 and 2 blues whose sum is 10 from box 2)
$= \frac{1}{2025} \times 12 = \frac{4}{675}$

7 (a) The points *a*, *b*, *c*, *d*, *e* and *f* lie on a circle.



(i) If these points are used as vertices, how many different quadrilateral can be formed?

(ii) How many of these quadrilaterals will have [*ab*] as one side?

7 (b) Three cards are drawn, at random and without replacement, from a pack of 52 playing cards. Find the probability that

(i) the three cards drawn are the Jack of clubs, the Queen of clubs and the King of clubs

(ii) the three cards are aces

(iii) two cards are black and one card is a diamond

(iv) the three cards are of the same colour.

7 (c) The mean of the real numbers *q*, *r*, *s* and *t* is $\bar{x}$ and the standard deviation is σ. Consider the numbers $\beta q+\alpha$, $\beta r+\alpha$, $\beta s+\alpha$ and $\beta t+\alpha$ where $\beta, \alpha \in \mathbf{R}$ and $\beta > 0$.

(i) Show that the mean of these numbers is $\beta \bar{x}+\alpha$.

(ii) Show that the standard deviation of these numbers is $\beta\sigma$.

SOLUTION

7 (a) (i)

How many ways can 4 points be picked from 6 points?

$${}^{6}C_{4}=\frac{6\times 5\times 4\times 3}{4\times 3\times 2\times 1}=15$$

The number of selections of *n* different objects taking *r* at a time $= {}^{n}C_{r}$ 11

7 (a) (ii)

If 2 points are already chosen, i.e. [*ab*], you need to find out how many ways you can pick the other 2 points from the remaining 4 points.

$${}^{4}C_{2}=\frac{4\times 3}{2\times 1}=6$$

7 (b) (i)

There are 52 cards in a pack.

There is a probability of $\frac{1}{52}$ of picking the Jack of Clubs on the first pick.

There is a probability of $\frac{1}{51}$ of picking the Queen of Clubs on your second pick, as there is one less card in the pack (non-replacement).

There is a probability of $\frac{1}{50}$ of picking the King of Clubs on your third pick.

However, you may not have picked them in that order. The number of arrangements of 3 objects, all taken, is 3!

p(Jack of Clubs and Queen of Clubs and King of Clubs) $= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times 3! = \frac{1}{22{,}100}$

7 (b) (ii)

There are 4 aces in a pack of 52.

The probability of picking an Ace on your first pick is $\frac{4}{52}$.

The probability of picking an Ace on your second pick is $\frac{3}{51}$ as there are 3 Aces left to be picked from 51 cards.

The probability of picking an Ace on your third pick is $\frac{2}{50}$ as there are 2 Aces left to be picked from 50 cards.

p(3 Aces) $= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{1}{5{,}525}$

[**Note**: You do not have to worry about the number of arrangements as all the picks are the same, i.e. aces.]

7 (b) (iii)

There are 26 black cards made up of 13 spades and 13 clubs. There are 26 red cards made up of 13 diamonds and 13 hearts.

The probability of picking a black card on your first pick is $\frac{26}{52}$.

The probability of picking a black card on your second pick is $\frac{25}{51}$.

The probability of picking a diamond on your third pick is $\frac{13}{50}$.

However, you may not have picked them in that order. The number of arrangements of 3 objects, all taken, 2 alike is $\frac{3!}{2!}$.

p(Black card and a Black card and a Diamond) $= \frac{26}{52} \times \frac{25}{51} \times \frac{13}{50} \times \frac{3!}{2!} = \frac{13}{68}$

7 (b) (iv)

p(3 Blacks) or p(3Reds) $= \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} + \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} = \frac{4}{17}$

7 (c) (i)

$$\bar{x} = \frac{x_1 + x_2 + \dots\dots\dots\dots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ 3

Numbers: q, r, s, t

$$\therefore \bar{x} = \frac{q+r+s+t}{4}$$

Numbers: $\beta q+\alpha,\ \beta r+\alpha,\ \beta s+\alpha,\ \beta t+\alpha$

$$\therefore \text{Mean} = \frac{\beta q+\alpha+\beta r+\alpha+\beta s+\alpha+\beta t+\alpha}{4} = \frac{\beta(q+r+s+t)+4\alpha}{4}$$

$$\therefore \text{Mean} = \frac{\beta(q+r+s+t)}{4} + \alpha = \beta\bar{x}+\alpha$$

7 (c) (ii)

x	d	d^2
q	$(q-\bar{x})$	$(q-\bar{x})^2$
r	$(r-\bar{x})$	$(r-\bar{x})^2$
s	$(s-\bar{x})$	$(s-\bar{x})^2$
t	$(t-\bar{x})$	$(t-\bar{x})^2$

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$ 6

where deviation $d = (x-\bar{x}) = (\text{Number} - \text{Mean})$

$$\sigma = \sqrt{\frac{(q-\bar{x})^2+(r-\bar{x})^2+(s-\bar{x})^2+(t-\bar{x})^2}{4}}$$

x	d	d^2
$\beta q+\alpha$	$\beta(q-\bar{x})$	$\beta^2(q-\bar{x})^2$
$\beta r+\alpha$	$\beta(r-\bar{x})$	$\beta^2(r-\bar{x})^2$
$\beta s+\alpha$	$\beta(s-\bar{x})$	$\beta^2(s-\bar{x})^2$
$\beta t+\alpha$	$\beta(t-\bar{x})$	$\beta^2(t-\bar{x})^2$

$\leftarrow d = \beta q+\alpha-(\beta\bar{x}+\alpha) = \beta q+\alpha-\beta\bar{x}-\alpha$

$\therefore d = \beta(q-\bar{x})$

$$\text{Standard Deviation} = \sqrt{\frac{\beta^2(q-\bar{x})^2+\beta^2(r-\bar{x})^2+\beta^2(s-\bar{x})^2+\beta^2(t-\bar{x})^2}{4}}$$

$$= \sqrt{\frac{\beta^2[(q-\bar{x})^2+(r-\bar{x})^2+(s-\bar{x})^2+(t-\bar{x})^2]}{4}}$$

$$= \beta\sqrt{\frac{(q-\bar{x})^2+(r-\bar{x})^2+(s-\bar{x})^2+(t-\bar{x})^2}{4}}$$

$$= \beta\sigma$$