Discrete Maths (Q 6 & 7, Paper 2)

1998

6 (a) In how many ways can the letters of the word IRELAND be arranged if each letter is used exactly once in each arrangement?
In how many of these arrangements do the three vowels come together?

(b) Solve the difference equation

$$4u_{n+2} - 25u_{n+1} - 29u_n = 0, \text{ where } n \geq 0,$$

given that $u_0 = 0$ and $u_1 = 16 \cdot 5$.

(c) On an unbiased die, the numbers 1, 3 and 4 are coloured red and the numbers 2, 5 and 6 are coloured black.

(i) The die is thrown once. Find the probability of getting an even number or a red number.

(ii) The die is thrown three times with the following outcome:
the second throw shows a red number and the sum of the numbers on the first and second throws is equal to the number on the third throw.
Find the probability of this outcome.

Solution

6 (a)

There are 6 ways to arrange the first letter.
Once this letter is in place there are 5 ways to arrange the next letter and so on.

> The number of arrangements of *n* different objects all taken, no repeats = *n*! **8**

No. of arrangements = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 7! = 5040

Glue the 3 vowels together and treat it as one letter. Have a look at one such arrangement.

R	L	N	D	IEA

There are 5! ways of arranging the 5 boxes. There are also 3! ways of arranging the vowels in the last box.

No. of arrangements = $5! \times 3! = 120 \times 6 = 720$

6 (b)

$$u_n = l(\alpha)^n + m(\beta)^n \quad \ldots\ldots\ (1)$$

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $4u_{n+2} - 25u_{n+1} - 29u_n = 0$

2. $4x^2 - 25x - 29 = 0$

3. $(4x - 29)(x + 1) = 0$

$\therefore \alpha = \frac{29}{4}, \ \beta = -1$

4. $u_n = l(\frac{29}{4})^n + m(-1)^n$

5. $u_0 = 0 \Rightarrow l + m = 0 \ldots\ldots(\mathbf{1})$

$u_1 = 16.5 \Rightarrow l(\frac{29}{4}) + m(-1) = 16.5 \Rightarrow 29l - 4m = 66 \ldots\ldots(\mathbf{2})$

Solve Equations (**1**) and (**2**) simultaneously.

$$\begin{aligned} l + m &= 0 \ldots\ldots(\mathbf{1})(\times 4) \\ 29l - 4m &= 66 \ldots(\mathbf{2}) \end{aligned} \quad \rightarrow \quad \begin{aligned} 4l + 4m &= 0 \\ 29l - 4m &= 66 \\ \hline 33l \quad &= 66 \Rightarrow l = 2 \end{aligned}$$

Substitute this value of l into Eqn. (**1**):

$\Rightarrow 2 + m = 0 \Rightarrow m = -2$

Answer: $u_n = 2(\frac{29}{4})^n - 2(-1)^n$

6 (c) (i)

EITHER OR Rule

The probability that either one event or another will occur is the sum of their separate probabilities minus the probability of the elements common to the events.

$$p(\text{Either } E_1 \text{ or } E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) \quad \ldots\ldots\ (13)$$

1	3	4
2	5	6

p(Either an Even or a Red) = p(Even) + p(Red) – p(Even Red) $= \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6}$

6 (c) (i)

In these situations write out all the possibilities:

1 3 4

2 5 6

1st. throw		2nd. throw		3rd. throw
1	+	1	=	2
1	+	3	=	4
1	+	4	=	5
2	+	1	=	3
2	+	3	=	5
2	+	4	=	6
3	+	1	=	4
3	+	3	=	6
4	+	1	=	5
5	+	1	=	6

There are 10 possibilities. Work out the probability of the first possibility and multiply your answer by 10.

$p(1 \text{ and then } 1 \text{ and then } 2) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

p(Second throw shows a red number and the sum of the numbers on the first and second throws is equal to the number on the third throw) $= \frac{1}{216} \times 10 = \frac{5}{108}$

7 (a) If p is the mean of the numbers a, b, c, d express in terms of p and k the mean of the numbers $2a + k$, $2b + k$, $2c + k$, $2d + k$.

(b) A classroom contains 15 desks which are arranged in rows.
The front row contains 3 desks.
15 students are seated at random in the classroom, 8 of whom are boys and 7 of whom are girls.
Each desk seats only one student.
What is the probability that

(i) three girls occupy the front row of desks?

(ii) there are more boys than girls seated in the front row?

(iii) there are two girls and one boy seated in the front row with the two girls seated next to each other?

(c) The numbers p, q, r have a mean $\bar{x}$ and a standard deviation σ.

(i) Express $\bar{x}$ in terms of p, q and r.

(ii) Show that

$$\sigma^2 = \tfrac{1}{3}(p^2 + q^2 + r^2) - (\bar{x})^2.$$

Solution

7 (a)

$$\bar{x} = \frac{x_1 + x_2 + \dots\dots\dots\dots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N} \quad \dots\dots\dots \quad (3)$$

Numbers: a, b, c, d

Mean $p = \dfrac{a+b+c+d}{4}$

Numbers: $2a+k,\ 2b+k,\ 2c+k,\ 2d+k$

Mean $\bar{x} = \dfrac{2a+k+2b+k+2c+k+2d+k}{4}$

$\Rightarrow \bar{x} = \dfrac{2(a+b+c+d)+4k}{4} = 2\left(\dfrac{a+b+c+d}{4}\right)+k$

$\therefore \bar{x} = 2p+k$

7 (b) (i)

Forget about the desks. There are 15 students which includes 8 boys and 7 girls. You are asked to randomly pick 3 students.

The probability of choosing a girl on your first pick is $\frac{7}{15}$ as there are 7 girls to pick from 15 students.

The probability of choosing a girl on your second pick is $\frac{6}{14}$ as there are now 6 girls to pick from 14 students.

The probability of choosing a girl on your third pick is $\frac{5}{13}$ as there are now 5 girls to pick from 13 students.

$p(\text{3 girls}) = \frac{7}{15} \times \frac{6}{14} \times \frac{5}{13} = \frac{1}{13}$

7 (b) (ii)

More boys than girls means you can choose 3 boys and no girls OR 2 boys and 1 girl. Work out each probability and add them together as OR means add.
Using the same reasoning as in part **(i)** you can work out the probability of randomly pick 3 boys.

$p(\text{3 boys}) = \frac{8}{15} \times \frac{7}{14} \times \frac{6}{13} = \frac{8}{65}$

Now work out the probability of randomly choosing 2 boys and a girl:

The probability of choosing a boy on your first pick is $\frac{8}{15}$.

The probability of choosing a boy on your second pick is $\frac{7}{14}$.

The probability of choosing a girl on your third pick is $\frac{7}{13}$ as there are 7 girls to pick from 13 students.
However, you may not have picked them in that order. So you need to multiply the probability by the number of ways of arranging 3 objects in which 2 are alike.

Number of arrangements of n objects, p alike of one kind, q alike of another kind $= \dfrac{n!}{p!q!}$ …… (10)

$p(\text{2 boys and 1 girl}) = \frac{8}{15}\times\frac{7}{14}\times\frac{7}{13}\times\frac{3!}{2!} = \frac{28}{65}$

$p(\text{3 boys})$ OR $p(\text{2 boys and 1 girl}) = \frac{8}{15}\times\frac{7}{14}\times\frac{6}{13} + \frac{8}{15}\times\frac{7}{14}\times\frac{7}{13}\times\frac{3!}{2!} = \frac{36}{65}$

7 (b) (iii)

The order in which you choose a boy or girl is important. There are 2 possibilities in which 2 girls are chosen together.

p(Girl and then Girl and then Boy) OR p(Boy and then Girl and then Girl)

$= \frac{7}{15}\times\frac{6}{14}\times\frac{8}{13} + \frac{8}{15}\times\frac{7}{14}\times\frac{6}{13} = \frac{16}{65}$

7 (c) (i)

$$\bar{x} = \frac{x_1 + x_2 + \dots\dots\dots\dots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ 3

$$\bar{x} = \frac{p+q+r}{3}$$

7 (c) (ii)

x	d	d^2
p	$(p-\bar{x})$	$(p-\bar{x})^2$
q	$(q-\bar{x})$	$(q-\bar{x})^2$
r	$(r-\bar{x})$	$(r-\bar{x})^2$

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$ 6

where deviation $d = (x-\bar{x}) = (\text{Number} - \text{Mean})$

$$\sigma = \sqrt{\frac{(p-\bar{x})^2 + (p-\bar{x})^2 + (p-\bar{x})^2}{3}}$$

$$\Rightarrow \sigma^2 = \frac{(p-\bar{x})^2 + (p-\bar{x})^2 + (p-\bar{x})^2}{3}$$

$$\Rightarrow \sigma^2 = \frac{p^2 - 2p\bar{x} + (\bar{x})^2 + q^2 - 2q\bar{x} + (\bar{x})^2 + r^2 - 2r\bar{x} + (\bar{x})^2}{3}$$

$$\Rightarrow \sigma^2 = \frac{p^2 + q^2 + r^2 - 2\bar{x}(p+q+r) + 3(\bar{x})^2}{3}$$

$$\Rightarrow \sigma^2 = \frac{p^2 + q^2 + r^2}{3} - 2\bar{x}\frac{(p+q+r)}{3} + \frac{3(\bar{x})^2}{3}$$

$$\Rightarrow \sigma^2 = \frac{p^2 + q^2 + r^2}{3} - 2\bar{x}(\bar{x}) + (\bar{x})^2$$

$$\Rightarrow \sigma^2 = \frac{p^2 + q^2 + r^2}{3} - 2(\bar{x})^2 + (\bar{x})^2$$

$$\therefore \sigma^2 = \tfrac{1}{3}(p^2 + q^2 + r^2) - (\bar{x})^2$$