1996

6 (a) In how many ways can a group of five be selected from ten people?
How many groups can be selected if two particular people from the ten cannot be in the same group?

(b) There are seven white and four black beads in a bag. A bead is picked at random and not replaced. A second bead is then picked.
(i) Find the probability that both beads are the same colour.

The two beads are returned to the bag and three red beads are added. Three beads are then picked at random without replacement. Find the probability that
(ii) all three beads are different in colour

(iii) at least two beads of the same colour are picked.

(c) Show that

$u_n = \frac{1}{3}\{(1+\sqrt{3})^n - (1-\sqrt{3})^n\}$

is the solution of the difference equation

$u_{n+2} - 2u_{n+1} - 2u_n = 0,\ n \geq 0$

when $u_0 = 0$ and $u_1 = \frac{2\sqrt{3}}{3}$.

Verify this solution.

SOLUTION

6 (a)

Number of ways in which 5 people can be selected from 10 people:

> The number of selections of n different objects taking r at a time $= {}^nC_r$ 11

$${}^{10}C_5 = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} = 252$$

Let's say that John and Mary cannot be together in the same group. Work out the number of groups in which they are together and subtract this number from 252.

Number of groups in which John and Mary are selected, i.e. you now need to select 3 people from 8 people:

$${}^8C_3 = \frac{8\times7\times6}{3\times2\times1} = 56$$

Number of groups in which 2 particular people (John and Mary) cannot be together:

$252 - 56 = 196$

6 (b) (i)

7 White 4 Black

Both beads are the same colour means either picking 2 whites or 2 blacks. Work out each probability separately and add them together.

The probability of choosing a white bead on your first pick is $\frac{7}{11}$ as there are 7 white beads to pick from a total of 11 beads.

The probability of picking a white bead on your second pick is $\frac{6}{10}$ and there are only 6 white beads left to pick from the 10 remaining beads.
Carry out the same procedure when picking 2 black beads.

$p(\text{2 Whites or 2 blacks}) = \frac{7}{11}\times\frac{6}{10}+\frac{4}{11}\times\frac{3}{10}=\frac{27}{55}$

6 (b) (ii)

7 White 4 Black 3 Red

Work out the probability of picking a white bead followed by a black bead followed by a red bead and then multiply this answer by the number of arrangements of the 3 colours.

$p(\text{White and then a Black and then a Red}) = \frac{7}{14}\times\frac{4}{13}\times\frac{3}{12}=\frac{1}{26}$
However, you may not have picked them in that order so you need to multiply by the number of arrangements of 3 different colours.
$p(\text{3 different colours}) = \frac{7}{14}\times\frac{4}{13}\times\frac{3}{12}\times 3! = \frac{3}{13}$

6 (b) (iii)

At least one = 1 – p(None) At least two = 1 – p(None or one) etc...

...... **15**

$p(\text{At least 2 beads have the same colour}) = 1 - p(\text{None have the same colour}) = 1-\frac{3}{13}=\frac{10}{13}$

6 (c)

$u_n = l(\alpha)^n + m(\beta)^n$ **1**

Steps

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2}+qu_{n+1}+ru_n=0$
2. Write down the corresponding quadratic equation: $px^2+qx+r=0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $u_{n+2} - 2u_{n+1} - 2u_n = 0$

2. $x^2 - 2x - 2 = 0$

3. $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \dfrac{2 \pm \sqrt{12}}{2} = \dfrac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$\therefore \alpha = 1 + \sqrt{3},\ \beta = 1 - \sqrt{3}$

4. $u_n = l(1+\sqrt{3})^n + m(1-\sqrt{3})^n$

5. $u_0 = 0 \Rightarrow l + m = 0 \Rightarrow m = -l$....(**1**)

$u_1 = \frac{2\sqrt{3}}{3} \Rightarrow l(1+\sqrt{3}) + m(1-\sqrt{3}) = \frac{2\sqrt{3}}{3}$....(**2**)

Solve Eqns. (**1**) and (**2**) simultaneously to find l and m:

$m = -l \Rightarrow l(1+\sqrt{3}) - l(1-\sqrt{3}) = \frac{2\sqrt{3}}{3}$

$\Rightarrow l + \sqrt{3}l - l + \sqrt{3}l = \frac{2\sqrt{3}}{3}$

$\Rightarrow 2\sqrt{3}l = \frac{2\sqrt{3}}{3}$

$\therefore l = \frac{1}{3} \Rightarrow m = -\frac{1}{3}$

Answer: $u_n = \frac{1}{3}(1+\sqrt{3})^n - \frac{1}{3}(1-\sqrt{3})^n = \frac{1}{3}\{(1+\sqrt{3})^n - (1-\sqrt{3})^n\}$

Checking your solution:
In the difference equation, let $n = 0$.

$\therefore u_2 - 2u_1 - 2u_0 = 0$

$\Rightarrow u_2 = 2u_1 + 2u_0$

$\Rightarrow u_2 = 2(0) + 2(\frac{2\sqrt{3}}{3})$

$\therefore u_2 = \frac{4\sqrt{3}}{3}$

Show that you get the same value for u_2 using your solution for u_n.

$u_2 = \frac{1}{3}\{(1+\sqrt{3})^2 - (1-\sqrt{3})^2\}$

$\Rightarrow u_2 = \frac{1}{3}\{1 + 2\sqrt{3} + 3 - 1 + 2\sqrt{3} - 3\}$

$\Rightarrow u_2 = \frac{1}{3}\{4\sqrt{3}\}$

$\therefore u_2 = \frac{4\sqrt{3}}{3}$

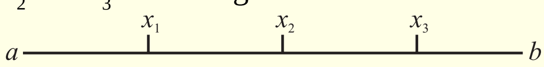
7 (a) Four numbers have a mean p.
Five numbers have a mean x.
These nine numbers have a mean q.
Express x in terms of p and q.

(b) Two dice A and B are cast. What is the probability of getting
(i) a total of two or a total of six?

(ii) a total greater than nine or a total which is prime?

(iii) a total which is three times as great as other possible totals?

(c) Real numbers x_1, x_2 and x_3 are each greater than a and less than b as shown on the number line.



Prove that
(i) $a < \bar{x} < b$ where $\bar{x}$ is the mean of x_1, x_2 and x_3.

(ii) $\sigma \le b - a$ where σ is the standard deviation of x_1, x_2 and x_3.

Solution

7 (a)

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N} \quad \ldots\ldots\ (3)$$

4 Numbers: $x_1,\ x_2,\ x_3,\ x_4$

$$p = \frac{x_1 + x_2 + x_3 + x_4}{4} \Rightarrow 4p = x_1 + x_2 + x_3 + x_4$$

5 Numbers: $x_5,\ x_6,\ x_7,\ x_8,\ x_9$

$$x = \frac{x_5 + x_6 + x_7 + x_8 + x_9}{5} \Rightarrow 5x = x_5 + x_6 + x_7 + x_8 + x_9$$

9 Numbers: $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9$

$$q = \frac{(x_1 + x_2 + x_3 + x_4) + (x_5 + x_6 + x_7 + x_8 + x_9)}{9}$$

$$\Rightarrow q = \frac{4p + 5x}{9} \Rightarrow 9q = 4p + 5x$$

$$\Rightarrow 5x = 9q - 4p$$

$$\therefore x = \frac{9q - 4p}{5}$$

7 (b) (i)

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)}$$ **12**

Total number of possible outcomes = 36

$p(\text{Total of 2 or 6}) = \frac{6}{36} = \frac{1}{6}$

7 (b) (ii)

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Prime Numbers: 2, 3, 5, 7, 11

$p(\text{Total of a prime number or greater than 9}) = \frac{19}{36}$

7 (b) (iii)

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Rolling 2 sixes gives a total of 12 which is three times greater than say the total rolled by a 1 and 2.

$p(\text{Total which is 3 times as great as other possible totals}) = \frac{10}{36} = \frac{5}{18}$

7 (c) (i)

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ **3**

$a < x_1 < b$

$a < x_2 < b$

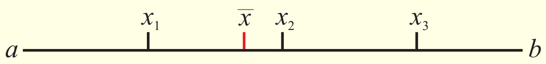
$a < x_3 < b$

$3a < x_1 + x_2 + x_3 < 3b$

$$\Rightarrow a < \frac{x_1 + x_2 + x_3}{3} < b$$

$\therefore a < \bar{x} < b$

7 (c) (ii)



You have proved in part (**i**) that $\overline{x}$ lies on the number line between *a* and *b*.
You can see from the number line that the following statements are true:

$$|x_1 - \overline{x}| < |b - a| \Rightarrow (x_1 - \overline{x}) < (b-a)^2$$

$$|x_2 - \overline{x}| < |b - a| \Rightarrow (x_2 - \overline{x}) < (b-a)^2$$

$$|x_3 - \overline{x}| < |b - a| \Rightarrow (x_3 - \overline{x}) < (b-a)^2$$

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}} \quad \ldots\ldots\ \textbf{6}$$

where deviation $d = (x - \overline{x}) = (\text{Number} - \text{Mean})$

$$\sigma = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2}{3}} < \sqrt{\frac{(b-a)^2 + (b-a)^2 + (b-a)^2}{3}}$$

$$\Rightarrow \sigma < \sqrt{\frac{\cancel{3}(b-a)^2}{\cancel{3}}}$$

$$\therefore \sigma < b - a$$